Method of Overlapping Patches for Electromagnetic Computation Near Imperfectly Conducting Cusps and Edges

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***Abstract*—An asymptotic solution method for three-dimensional electromagnetic field problem inside and outside an imperfectly conducting cusp is proposed. Implementation of the method leads to a boundary value problem with singular boundary conditions. This problem can be solved numerically using the partition of unity method. Wave propagation in an L-shaped waveguide is modeled as an example.**

***Index Terms*—Asymptotic solutions, finite elements (FEs), impedance boundary conditions, partition of unity.**

I. INTRODUCTION

THE PROBLEM of electromagnetic wave propagation through compound structures that comprise two or more different homogeneous materials with parallel interfaces is completely (at least theoretically) solved. In contrast, propagation of the wave in the structure that consists of several components with nonparallel surfaces is by no means trivial. To the best of the authors' knowledge, even the simplest problem of that kind, the problem of diffraction on a conducting wedge, is still waiting for a complete solution.

Let us start from some functional analysis aspects of the problem. Obviously, the energy of the system must be finite. That is, components of electric $\vec{E}$ and magnetic $\vec{H}$ fields must belong to the function space $H^1_{\text{curl}} = \{\vec{v} \in L_2(\Omega)^3\text{: } \nabla \times \vec{v} \in L_2(\Omega)^3\}$. Note that if a vector function belongs to H^1_{curl}, it does not follow that its components belong to the space $H^1_{12} \equiv \{f \in L_2(\Omega)\text{: } D^1 f \in L_2\}$. So, in general, these components can be square-integrable but singular functions. In fact, this is the case when an electromagnetic wave is scattered by an edge or cusp of a good conductor. The present paper is devoted to the study of this situation.

In Section II, the general idea of the method of surface impedance is described. In Section III we propose a general three-dimensional (3-D) formulation of a new asymptotic approach to the analysis of the behavior of the electromagnetic field in the vicinity of an edge or cusp of a good conductor. In Section IV the method is implemented for the case of a conducting edge.

It has been shown [1] that at the edge of a good conductor the surface impedance experiences a singularity. Hence a numerical approximation taking into account the singularity needs to be constructed. To do so, we employ the Partition of Unity Method (PUM) on overlapping patches (Section V).

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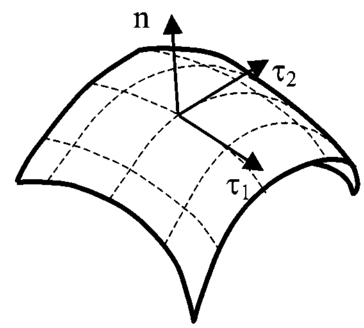

Fig. 1. Local coordinates on the surface of the conductor.

II. THE METHOD OF SURFACE IMPEDANCE

When modeling diffraction of a monochromatic electromagnetic wave on the edge of a good conductor, one normally needs to know the field distribution outside the conductor. So instead of solving the Maxwell equations in the whole domain, we would like to formulate the problem in the exterior region substituting the conductor with corresponding boundary conditions. Usually, these conditions can be written in the form

$$\left. \left(E_{\tau_i} + Z_j(\tau_1, \tau_2) \frac{\partial E_{\tau_i}}{\partial \vec{n}} \right) \right|_\Gamma = 0. \quad (1)$$

In (1)

$$Z_{ij} = \left. \frac{E_{\tau_i}}{H_{\tau_j}} \right|_\Gamma$$

is the impedance, the ratio of the tangential components of electric and magnetic fields, and the set $\{\tau_1, \tau_2, n\}$ forms a local orthogonal (generally speaking, curvilinear) coordinate system (Fig. 1).

With the boundary conditions of type (1), we arrive at a well-posed boundary value problem in the region outside the conductor and could solve it by any standard numerical or (possibly) analytical method.

III. The Perturbation Method

To approximate the impedance in the vicinity of a conducting edge, we would like to find a perturbation technique that is applicable in this area.

Let us return to the system of Maxwell equations. As a first step, we switch to nondimensional variables [2] by introducing appropriate scaling factors for the field

$$\tilde{\vec{H}} = (I/4\pi\lambda)^{-1}\vec{H}, \quad \tilde{\vec{E}} = (\mu I\omega/8\pi)^{-1}\vec{E} \tag{2.a}$$

and coordinates

$$\chi_i = \frac{x_i}{\delta} \tag{2.b}$$

where, I is a characteristic current, λ is the wavelength, ω is the frequency of the incident wave, $\delta = (1/\omega\sigma\mu)^{1/2}$ is the skin depth, and x_i is a corresponding space coordinate (r in the case of the polar coordinate system).

We can now rewrite Maxwell's equations in the conductor in a nondimensional form:

$$p\nabla \times \tilde{\vec{H}} = \tilde{\vec{E}} \quad \nabla \times \tilde{\vec{E}} = -ip\tilde{\vec{H}} \tag{3}$$

where parameter p is defined as $p = \delta/\lambda$. We assume that $p = o(1)$, which corresponds to the case of diffraction on a *good* conductor.

Since the ratio $p = \delta/\lambda$ is the only nondimensional parameter of the problem, it is natural to seek the solution of the system (3) as an asymptotic series in this small parameter

$$\tilde{\vec{E}} = \sum_n p^{\gamma+n}\tilde{\vec{E}}^{(n)} \quad \tilde{\vec{H}} = \sum_n p^{\gamma+n}\tilde{\vec{H}}^{(n)}. \tag{4}$$

Parameter γ in (4) is defined by the continuity conditions on the conductor-dielectric interface and the zero-order approximation of the field in the outer vicinity of the conductor. This approximation corresponds to the solution for the perfect conductor. Substituting (4) into (3), we come to the chain of equations

$$\tilde{\vec{E}}^{(0)} = 0, \; \tilde{\vec{E}}^{(n)} = \nabla \times \tilde{\vec{H}}^{(n-1)}$$
$$\nabla \times \nabla \times \tilde{\vec{H}}^{(n-1)} + i\tilde{\vec{H}}^{(n-1)} = 0. \tag{5}$$

Finally, the asymptotic expansions (4) of the electromagnetic field in a good conductor lead to the following expression for the first nonvanishing term in the asymptotic expansion of the surface impedance:

$$Z_{ij} = pE^{(1)}_{\tau_i} \Big/ H^{(0)}_{\tau_j}\Big|_{\Gamma}. \tag{6}$$

Summarizing, we have the following algorithm for the asymptotic approximation of the impedance.

- Find the distribution of the magnetic field in the outer vicinity of the perfect conductor.

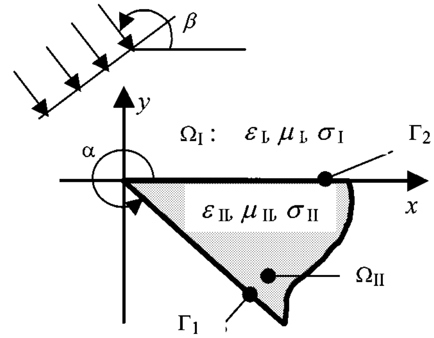

Fig. 2. Conducting edge.

- Using boundary conditions generated by the solution of the previous problem, find the first nonvanishing term $\tilde{\vec{H}}^{(0)}$in the asymptotic expansion for the magnetic field in the conductor

$$\nabla \times \nabla \times \tilde{\vec{H}}^{(0)} + i\tilde{\vec{H}}^{(0)} = 0 \quad \tilde{\vec{H}}_{\tau}\Big|_{\Gamma} = \vec{H}^{\text{dielectric}}_{\tau}\Big|_{\Gamma};$$
$$\tilde{\vec{H}}\Big|_{\infty} = 0. \tag{7}$$

- Using the solution of (6), find the first nonvanishing term $\tilde{\vec{E}}^{(1)}$ in the asymptotic expansion for the electric field in the conductor

$$\tilde{\vec{E}}^{(1)} = \nabla \times \tilde{\vec{H}}^{(0)}.$$

- Finally, using (6) and the results of the two preceding items, find the surface impedance.

IV. Diffraction on a Two-Dimensional Conducting Edge

Let us implement the program introduced in Section III in the particular case of diffraction of a plane wave on a conducting edge. We assume that the electric part of the incident field has the only nonzero component that is parallel to the edge (Fig. 2). Let us take this direction as the z axis of a cylindrical coordinate system (r, φ, z). We assume that the medium surrounding the conductor is characterized by constants $\varepsilon_I, \mu_I, \sigma_I$. Conductivity σ_I is assumed to be *positive* but arbitrarily small. Since $\sigma_I \neq 0$, in the whole domain surrounding the conductor $\nabla \cdot E = 0$. Taking into account that the electric field possesses only the z component, we conclude that $\partial E_z/\partial z = 0$. That is, the field does not depend on z. The same is true for the magnetic component $\vec{H}$of the wave.

The first item of the plan has been carried out in [3]: the electric field in the vicinity of a perfect conductor is given by the formula

$$E_I(r, \varphi) = \frac{i\omega}{c}\sum_{m=1}^{\infty}\sin\left(\pi m\frac{\varphi}{\alpha}\right)\sin\left(\pi m\frac{\beta}{\alpha}\right)J_{\pi m/\alpha}(k_I r). \tag{8}$$

Now, we turn to the next two items of our plan. We need to reconstruct the electromagnetic field in the conducting area Ω_{II} (see Fig. 2), given the tangential components of the magnetic field on its boundaries. In the general case of an arbitrary angle α, this can be done by solving the following boundary value problem:

$$\Delta E_z^{(1)} + k_{II}^2 E_z^{(1)} = 0$$
$$\left.\frac{\partial E_z}{\partial \varphi}\right|_\Gamma = H_r^{\text{diel}}\big|_\Gamma \; \left.\frac{\partial E_z}{\partial \varphi}\right|_\infty = 0 \quad \varphi \in [\alpha, 2\pi] \qquad (9)$$

where $H_r^{\text{diel}}|_\Gamma$ is the tangential component of the magnetic field on the surface of the perfect conductor [that is defined by the electric field (8)] and $E_z^{(1)}$ is the first nonvanishing term in the series (4). Solution of this problem is known [1]

$$E_z^{(1)} = \sum_{m=0}^{\infty} \cos(\pi m/\beta(\varphi - \alpha)) \times \left\{ Y_{\nu_m}(k_{II} r) \int_0^{k_{II} r} d\xi\, \xi F(\xi) J_{\nu_m}(\xi) - J_{\nu_m}(k_{II} r) \int_{k_{II} r}^{\infty} d\xi\, \xi F(\xi) Y_{\nu_m}(\xi) \right\} \qquad (10)$$

where $\beta = 2\pi - \alpha$, $\nu = \pi m/\beta$, $k_{II}^2 = -(i\omega/c^2)(i\omega + 4\pi\sigma_{II})$, and $J_\nu(z)$ and $Y_\nu(z)$ are Bessel and Weber functions of the νth order, respectively, and

$$F(\xi) = H_r^{\text{diel}}\big|_{\varphi=\alpha} - (-1)^m \, H_r^{\text{diel}}\big|_{\varphi=0}.$$

Since

$$\lim_{z \to 0} Y_\nu(z) J_\nu(z) = -\frac{\pi\nu}{\sin \pi\nu} \neq \infty$$

all integrals in (10) exist.

Equality (10), together with (8), completes the general procedure of defining the first nonvanishing asymptotic term of the impedance. Results of the asymptotic analysis of (10) are given in [1].

V. Numerical Example: Linearly Polarized EM Wave in a Γ-Shaped Waveguide With Imperfectly Conducting Walls

Let us consider propagation of a linearly polarized wave in a Γ-shaped waveguide. We assume that the only component of the electrical field is directed along the z axis (Fig. 3); Γ_1 and Γ_2 are surfaces of perfect and imperfect conductors, respectively, and Γ_0 is the waveguide port. The problem can be formulated in terms of the electrical component E_z of the wave

$$\nabla^2 E_z + k^2 E_z = 0 \quad E_z|_{\Gamma_0} = E_0 \sin(k_y y + \varphi_0);$$
$$\left.\left(E_z + Z\frac{\partial E_z}{\partial \vec{n}}\right)\right|_{\Gamma_1} = 0, \quad E_z|_{\Gamma_2} = 0. \qquad (11)$$

In (11), Z is the electric impedance whose behavior in the vicinity of the edge can be approximated as [1]

$$Z_x \equiv \left.\frac{E_z}{H_x}\right|_{\Gamma_1} \cong -\frac{\tan(\pi/6)}{\sigma\rho} \quad Z_y \equiv \left.\frac{E_z}{H_y}\right|_{\Gamma_1} \cong \frac{\tan(\pi/6)}{\sigma\rho}. \qquad (12)$$

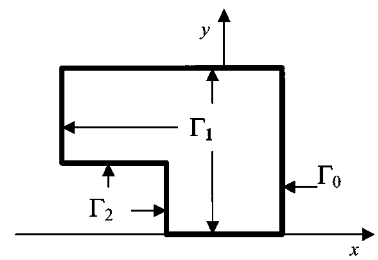

Fig. 3. Geometry of the problem.

We represent the electric field as the sum

$$E_z = E_0 + u \qquad (13)$$

where E_0 is any smooth function satisfying the boundary conditions

$$E_0|_{\Gamma_0} = E_0 \sin(k_y y + \varphi_0), \quad E_0|_{\Gamma_1 \cup \Gamma_2} = 0. \qquad (14)$$

The boundary value problem for u is obtained by substituting representation (13) into the problem (11), and taking into account boundary conditions (14)

$$\nabla^2 u + k^2 u = -\nabla^2 E_0 + k^2 E_0 \quad u|_{\Gamma_0} = 0, \quad u|_{\Gamma_1} = 0$$
$$\left.\left(u + Z\frac{\partial(u + E_0)}{\partial \vec{n}}\right)\right|_{\Gamma_2} = 0. \qquad (15)$$

The weak formulation of this problem is well known

$$\int_\Omega (\nabla u \nabla u' - k^2 u u')\, d\Omega - \int_{\Gamma_1} \frac{1}{Z} u u' dS = -\int_\Omega (\nabla E_0 \nabla u' - k^2 E_0 u')\, d\Omega. \qquad (16)$$

Here, u' is a "test" function from $H^1(\Omega)$ satisfying the essential boundary conditions on $\partial\Omega$.

Discretization of (16) can be obtained from the weak formulation in the standard way, by restricting both u and u' to the finite-dimensional space constructed by method of partition of unity [4]. This space is designed as follows:

- the computational domain is covered by overlapping patches $\{\Omega_k\}$;
- each patch Ω_k is endowed with a local system of approximating functions $\{g_k\}$. Functions g_k are chosen to provide a good local approximation for the solution over the patch. In particular, they can be singular.
- the numerical solution is sought as a linear combination of n functions ψ_i

$$u_{\text{approx}}(x) = \sum_{i=1}^{n} a_i \psi_i, \qquad \psi_i \equiv g_k \varphi_m$$

  that also act as trial functions u' in (16). Here, ψ_i is a local approximation function g_j weighted by the corresponding element of the partition of unity φ_m [4].

A detailed description of PUM and its implementation is given in our accompanying paper [5]. Here we concentrate on one aspect of the method. To effectively approximate the solution in patches adjacent to the edge we include in the corresponding set of approximating functions a singular but

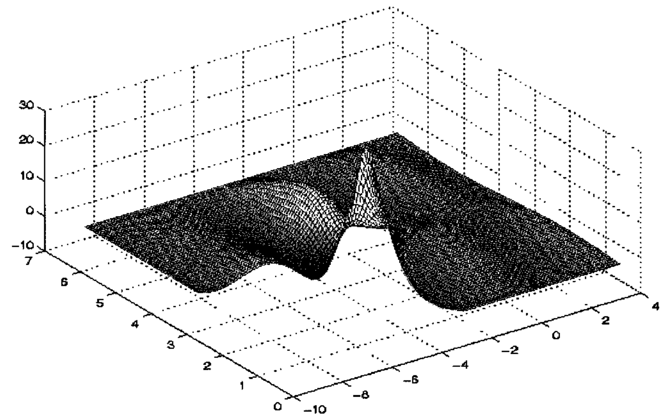

Fig. 4. Electric field in a waveguide with imperfectly conducting walls.

square-integrable function $v(x, y) = \rho^{-\gamma}$ (ρ being the distance from the edge, and $\gamma = 1/3$ in our particular case). Since the solution is unbounded in the vicinity of the singularity point, the *absolute* value of the numerical error ε can be expected to be very large for virtually any numerical method used. It therefore makes more practical sense to evaluate the *relative* error that is asymptotically $O(\varepsilon/\rho^{-\gamma})$. Measuring this relative error is qualitatively equivalent to using a weighted norm with the weight defined as follows. First, for every singular point (x_i, y_i) and corresponding patch Ω_n we introduce a function

$$w_{i,n} = \varphi_n(x, y) \cdot \{(\rho_i)^{2\gamma_i} - 1\} + 1 \quad (17)$$

where $\rho_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ is the distance from the ith singular point (x_i, y_i) in the patch Ω_n; $\gamma_i > 0$ is the order of the singularity, and $\varphi_n(x, y)$ is the element of the partition of unity whose value at the singular point is one and whose normal derivative on the boundary of the patch Ω_n is zero. Note that in (17), it is implicitly assumed that each singular point belongs to one patch.

Then the weight function w and the new W-norm are defined as

$$w = \prod_n w_n \quad (18.a)$$

$$\|u\|_W = \|uw\|_{L_2}. \quad (18.b)$$

Each approximation function satisfies the conditions

$$\|g_j\|_W < \infty \quad (19.a)$$

$$\|\nabla\lambda_j\|_W < \infty. \quad (19.b)$$

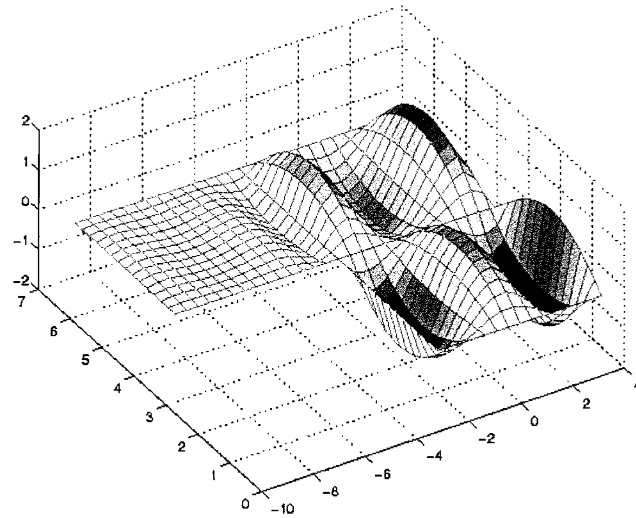

Fig. 5. Electric field in the waveguide with perfectly conducting walls.

The approximating functions g_k were taken as polynomials of order up to six

$$g_k = (x - x_0)^n (y - y_0)^{m-n}$$
$$0 \le m \le 6 \quad 0 \le n \le m. \quad (20)$$

Homogeneous boundary conditions on the port Γ_0 and on the perfectly conducting surface Γ_1 were imposed by a proper choice of constants x_0, y_0, and in each patch. Figs. 4 and 5 represent the numerical solutions obtained. In both cases, the domain is covered by 12 patches. The total number of degrees of freedom is 258.

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