High-order finite elements of complete and incomplete bases in electromagnetic-field computation

Z. Ren and N. Ida

Abstract: Function spaces of high-order differential-forms-based finite elements of complete bases are analysed and compared with those of incomplete bases. Elements of complete and incomplete bases differ only in the null space of the differential operator. They model the same range space of the differential operator. A comparison of 1-form 2nd order hierarchical elements (hierarchical edge elements) of complete and incomplete bases is carried out through the example of an eddy-current problem. It is found that there is no significant difference in the accuracy of the results given by the two kinds of elements, as long as the ratio mesh size/skin depth is sufficiently small. However elements of complete bases are much more expensive to use.

1 Introduction

Finite elements based on differential forms, namely the 1-form (edge) and 2-form (facet) elements, have proved their high efficiency in the computation of electromagnetic fields. The advantages of these elements are principally the capacity of allowing natural discretisation of the system with appropriate continuity of fields. For a given order of p-form elements, there exist two alternatives: the incomplete q-order bases [1, 2] and the complete q-order bases [3, 4]; both are complete to $q-1$ order under the differential operation. Whether to use the complete or incomplete order bases has been a subject of debate over the years.

In a previous paper [5], we gave a general description of high-order differential form-based elements. The relation of those element spaces, in particular the inclusion property, is clearly illustrated with the help of De Rham's complex. The analysis focused on elements of incomplete bases.

In this paper, the function spaces of high-order p-form elements of complete bases are analysed and compared with those of incomplete bases. After a short review of the function spaces of incomplete-order p-form elements, we give a description of function spaces of complete-order p-form elements, in particular, of the decomposition and the link of these spaces with the help of De Rham's complex. A comparison of the elements of complete and incomplete bases, namely the case of 1-form elements (one of the most useful elements in the computation of electromagnetic fields) is carried out. The performance of 2nd order 1-form hierarchical elements of complete and incomplete bases is compared through the example of an eddy-current problem.

IEE Proceedings online no. 20020252

DOI: 10.1049/ip-smt:20020252

Paper first received 4th October 2001 and in revised form 1st February 2002

Z. Ren is with the Simplex Solution Inc., 521 Almanor Ave., Sunnyvale, CA, 94085, USA

N. Ida is with the Department of Electrical Engineering, The University of Akron, Akron, OH, 44325-3904, USA

2 Function spaces of p-form elements of incomplete orders

Let W_q^p be the function space of incomplete q-order p-form elements constructed over a 3-simplex S^3 (a tetrahedron). The case of $q=1$ corresponds to the well known Whitney elements [2]. W_q^p can be decomposed into a null space of the differential operator Z_q^p (set of closed forms, i.e. forms vanishing under the differential operation) and a range space of the differential operator Y_q^p: $W_q^p = Z_q^p \oplus Y_q^p$.

In order to show how the element spaces are decomposed when the order is increased from $q-1$ to q, we introduce the following spaces of polynomials defined over S^3:

$$\tilde{P}_q = \text{linear space of homogenous polynomials of degree } q$$
$$\tilde{G}_q = \{\boldsymbol{v} \in (\tilde{P}_q)^3 | \boldsymbol{v} = \operatorname{grad}\phi,\ \phi \in \tilde{P}_{q+1}\},$$
$$\tilde{S}_q = \{\boldsymbol{v} \in (\tilde{P}_q)^3 | \boldsymbol{r}\cdot\boldsymbol{v} = 0\}$$

where $\boldsymbol{r}$ is the position vector.

These spaces are related by the Helmholtz decomposition: $(\tilde{P}_q)^3 = \tilde{G}_q \oplus \tilde{S}_q$. Their dimensions are, respectively,

$$\dim(\tilde{P}_q) = (q+1)(q+2)/2$$
$$\dim(\tilde{G}_q) = \dim(\tilde{P}_{q+1}) = (q+2)(q+3)/2$$
$$\dim(\tilde{S}_q) = q(q+2)$$

We define also:

$$\tilde{C}_q = \{\boldsymbol{v} \in (\tilde{P}_q)^3 | \boldsymbol{v} = \operatorname{curl}\boldsymbol{u}, \boldsymbol{u} \in \tilde{S}_{q+1}\}$$
$$\tilde{T}_q = \{\boldsymbol{v} \in (\tilde{P}_q)^3 | \boldsymbol{r}\times\boldsymbol{v} = 0\}$$

The space $(\tilde{P}_q)^3$ can also be decomposed to $(\tilde{P}_q)^3 = \tilde{C}_q \oplus \tilde{T}_q$. It has to be noted that the curl operator is an isomorphism of $\tilde{S}_{q+1}$ onto $\tilde{C}_q$, and the div operator is an isomorphism of $\tilde{T}_{q+1}$ onto $\tilde{P}_q$. We have, respectively,

$$\dim(\tilde{C}_q) = \dim(\tilde{S}_{q+1}) = (q+1)(q+3)$$
$$\dim(\tilde{T}_q) = \dim(\tilde{P}_{q-1}) = q(q+1)/2$$

We denote further by P_q the linear space of polynomials with degree up to q, i.e. $P_q = \tilde{P}_0 \oplus \ldots \oplus \tilde{P}_q$, and write $G_q = \tilde{G}_0 \oplus \cdots \oplus \tilde{G}_q$, $S_q = \tilde{S}_0 \oplus \cdots \oplus \tilde{S}_q$, $C_q = \tilde{C}_0 \oplus \cdots \oplus \tilde{C}_q$, $T_q = \tilde{T}_0 \oplus \cdots \oplus \tilde{T}_q$, respectively.

Using these definitions, the null spaces of p-form elements can be decomposed as ($q \geq 2$) [5]:

$$Z_1^0 = \tilde{P}_0 \quad Z_q^0 = Z_{q-1}^0 = P_0$$
$$Z_1^1 = \tilde{G}_0 \quad Z_q^1 = Z_{q-1}^1 \oplus \tilde{G}_{q-1} = G_{q-1}$$
$$Z_1^2 = \tilde{C}_0 \quad Z_q^2 = Z_{q-1}^2 \oplus \tilde{C}_{q-1} = C_{q-1}$$
$$Z_1^3 = \tilde{P}_0 \quad Z_q^3 = Z_{q-1}^3 \oplus \tilde{P}_{q-1} = P_{q-1}$$

and the range spaces are decomposed into

$$Y_1^0 = \tilde{P}_1 \quad Y_q^0 = Y_{q-1}^0 \oplus \tilde{P}_q$$
$$Y_1^1 = \tilde{S}_1 \quad Y_q^1 = Y_{q-1}^1 \oplus \tilde{S}_q = G_{q-1}$$
$$Y_1^2 = \tilde{C}_0 \quad Y_q^2 = Y_{q-1}^2 \oplus \tilde{T}_q = C_{q-1}$$
$$Y_1^3 = 0 \quad Y_q^3 = 0$$

Consequently, the function spaces of p-form elements have the decomposition ($q \geq 2$):

$$W_1^0 = P_1 \qquad W_q^0 = W_{q-1}^0 \oplus \tilde{P}_q = P_q$$
$$W_1^1 = \tilde{G}_0 \oplus \tilde{S}_1 \quad W_q^1 = W_{q-1}^1 \oplus \tilde{G}_{q-1} \oplus \tilde{S}_q = G_{q-1} \oplus S_q$$
$$W_1^2 = \tilde{C}_0 \oplus \tilde{T}_1 \quad W_q^2 = W_{q-1}^2 \oplus \tilde{C}_{q-1} \oplus \tilde{T}_q = C_{q-1} \oplus T_q$$
$$W_1^3 = \tilde{P}_0 \qquad W_q^3 = W_{q-1}^3 \oplus \tilde{P}_{q-1} = P_{q-1}$$

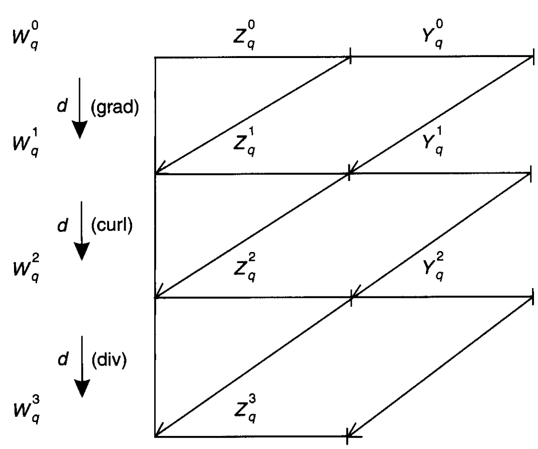

Fig. 1 *De Rham's complex showing the relation between p-form elements of incomplete bases*

The relation of these spaces can be illustrated with the help of De Rham's complex, Fig. 1, where d denotes the exterior differential operator in the calculus of differential forms. It represents the grad, curl and div operators of vector algebra. This diagram clearly shows the inclusion property of p-form elements. In particular, the differential operator is an isomorphism of Z_q^p onto Y_q^{p-1} and hence $\dim(Y_q^{p-1}) = \dim(Z_q^p)$.

The dimensions of the spaces Z_q^p, Y_q^p and W_q^p of p-form elements are summarised in Table 1.

3 Function spaces of p-form elements of complete orders

The q-order p-form elements W_q^p given in the previous section are complete to the $q-1$ order under the differential operation, but incomplete themselves to the q-order (except for 0-form elements), because their dimensions are smaller than that of a complete q-order vector or scalar basis. The reason is that the null spaces Z_q^p ($p = 1, 2, 3$) are only of $q-1$ order.

To get complete q-order element bases, it is sufficient to complete the null space Z_q^p to the q-order by adding, respectively, the gradient space $\tilde{G}_q$ to Z_q^1, the curl space $\tilde{C}_q$ to Z_q^2 and $\tilde{P}_q$ to Z_q^3. Let us denote by X_q^p the null spaces of the differential operator and by N_q^p the function spaces of complete q-order p-form elements. We have

$$X_q^0 = Z_q^0 \qquad X_q^1 = Z_q^1 \oplus \tilde{G}_q$$
$$X_q^2 = Z_q^2 \oplus \tilde{C}_q \quad X_q^3 = Z_q^3 \oplus \tilde{P}_q$$

and

$$N_q^0 = X_q^0 \oplus Y_q^0 \quad N_q^1 = X_q^1 \oplus Y_q^1$$
$$N_q^2 = X_q^2 \oplus Y_q^2 \quad N_q^3 = X_q^3$$

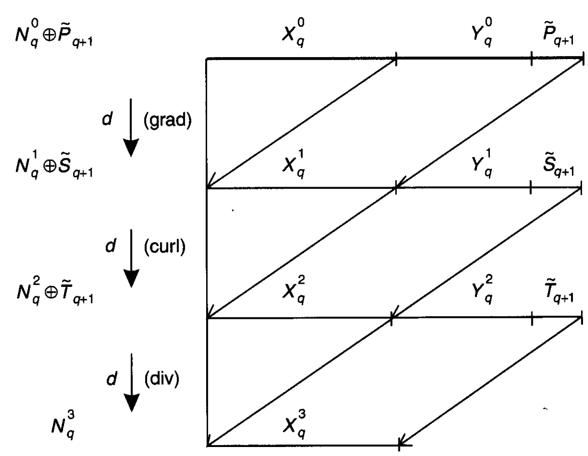

Fig. 2 *De Rham's complex showing the relation between p-form elements of complete bases*

Table 1: Dimension of function spaces of q-order p-form elements

Spaces Elements	Z_q^p (null space of d)	Y_q^p (range space of d)	W_q^p (function space)
0-form	1	$\frac{(q+1)(q+2)(q+3)-6}{6}$	$\frac{(q+1)(q+2)(q+3)}{6}$
1-form	$\frac{(q+1)(q+2)(q+3)-6}{6}$	$\frac{q(q+1)(2q+7)}{6}$	$\frac{q(q+2)(q+3)}{2}$
2-form	$\frac{q(q+1)(2q+7)}{6}$	$\frac{q(q+1)(q+2)}{6}$	$\frac{q(q+1)(q+3)}{2}$

Table 2: Dimension of function spaces of *q*-order *p*-form elements (complete basis)

Spaces Elements	X_q^p (null space of d)	Y_q^p (range space of d)	N_q^p (function space)
0-form	1	$\frac{(q+1)(q+2)(q+3)-6}{6}$	$\frac{(q+1)(q+2)(q+3)}{6}$
1-form	$\frac{(q+2)(q+3)(q+4)-6}{6}$	$\frac{q(q+1)(2q+7)}{6}$	$\frac{(q+1)(q+2)(q+3)}{2}$
2-form	$\frac{(q+1)(q+2)(2q+9)}{6}$	$\frac{q(q+1)(q+2)}{6}$	$\frac{(q+1)(q+2)(q+3)}{2}$
3-form	$\frac{(q+1)(q+2)(q+3)}{6}$	—	$\frac{(q+1)(q+2)(q+3)}{6}$

The relation of these spaces is illustrated in Fig. 2. with the help of De Rham's complex. It can be noticed that, contrary to the case of incomplete bases, the null space X_q^p of the *p*-form complete-order element is larger than the range space Y_q^{p-1} of the $p-1$ form element. Let us denote $\tilde{Q}_{q+1}^0 = \tilde{P}_{q+1}$, $\tilde{Q}_{q+1}^1 = \tilde{S}_{q+1}$ and $\tilde{Q}_{q+1}^2 = \tilde{T}_{q+1}$, respectively; the differential operator is an isomorphism of X_q^p onto $Y_q^{p-1} \oplus \tilde{Q}_{q+1}^{p-1}$ instead of onto Y_q^{p-1}.

The dimensions of the spaces X_q^p, Y_q^p and N_q^p of *p*-form elements are summarised in Table 2. It can be remarked that N_q^1 and N_q^2 are the family of vector elements given in [4]. The case of N_1^1 is the element presented earlier in [3].

Following the same analysis as presented in [5], the number of degrees of freedom to be assigned to each simplex can be determined. The cases of 1-form and 2-form elements are presented in Tables 3 and 4. Dimensions of the null and range spaces on each simplex are also given.

Comparing the spaces of *p*-form elements of complete bases with those of incomplete bases, it is seen that both are complete to the $q-1$ order under the differential operation. They model the same range space and differ only in the null space.

Consider the case of a 1-form (edge) element, the most commonly used vector element in the computation of electromagnetic fields. The incomplete and the complete bases differ only by a gradient space $\tilde{G}_q$. Adding $\tilde{G}_q$ to the function space of 1-form elements results only in additional irrotational functions. This will not contribute to the modelling of the range space of the curl operator and there will be no influence on the accuracy of the rotational fields. It is obvious that there is no need to use the complete order 1-form element to model the rotational field. This is particularly the case for a magnetostatic field problem using the vector potential formulation. However, the 1-form element of complete order allows a better approximation of the vector field itself. It might be useful for the modelling of the irrotational field, for example, in the case of eddy currents or high frequency problems.

The following Section compares 2nd order 1-form elements of complete and incomplete bases for eddy-current computation.

Table 3: Assignment of the degrees of freedom (DOF) of a complete-order 1-form (edge) element

Spaces DOF on	$X_q^1 = G_q$ null space of curl	$Y_q^1 = S_q$ range space of curl	N_q^1 function space
Edges	$6 \times q+3$	$6 \times 1-3$	$6 \times (q+1)$
Facets	$4 \times (q-1)q/2$	$4 \times (q-1)(q+2)/2$	$4 \times (q-1)(q+1)$
Volume	$(q-2)(q-1)q/6$	$(q-1)(q-2)(2q+3)/6$	$(q-2)(q-1)(q+1)/2$
Total	$(q+2)(q+3)(q+4)/6-1$	$q(q+1)(2q+7)/6$	$(q+1)(q+2)(q+3)/2$

Table 4: Assignment of the DOF of a complete-order 2-form (facet) element

Spaces DOF on	$X_q^2 = C_q$ null space of div	$Y_q^2 = T_q$ range space of div	N_q^2 function space
Facets	$4 \times q(q+3)/2+3$	$4 \times 1-3$	$4 \times (q+1)(q+2)/2$
Volume	$(q-1)q(2q+5)/6$	$q(q+1)(q+2)/6-1$	$(q-1)(q+1)(q+2)/2$

4 Comparison of complete and incomplete 2nd order 1-form elements

Hierarchical bases of 1-form (edge) elements of the complete and incomplete 2nd order [6], shown in Table 5, are applied in a formulation in terms of the magnetic vector potential–electric scalar potential to solve an eddy-current problem. The element of incomplete order W_2^1 takes the basis functions of the first three lines with 20 degrees of freedom. The complete basis takes the functions of the whole four lines with 30 degrees of freedom.

Table 5: Hierarchical 1-form 2nd order elements bases

Spaces			Edge functions DOF	Edge functions basis	Facet functions DOF	Facet functions basis
N_2^1	W_2^1	W_1^1	1×6	$\lambda_i d\lambda_j - \lambda_j d\lambda_i$		
		$\tilde{G}_1$	1×6	$d(\lambda_i \lambda_j)$		
		$\tilde{S}_2$			2×4	$\lambda_i(\lambda_j d\lambda_k - \lambda_k d\lambda_j)$
		$\tilde{G}_2$	1×6	$d(\lambda_i \lambda_j^2 - \lambda_j \lambda_i^2)$	1×4	$d(\lambda_j \lambda_j \lambda_k)$

The problem consists of calculating the eddy currents in a conductor inserted in the centre of the air gap of a magnetic circuit under sinusoidal excitation, Fig. 3, [6]. The study domain is meshed by 1600 tetrahedral elements. The number of unknowns is 10438 for the incomplete basis and 15414 for the complete basis. The equation system is

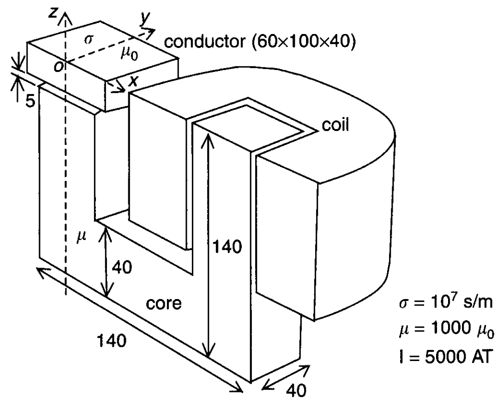

Fig. 3 *Example of an eddy-current problem, dimensions in mm*

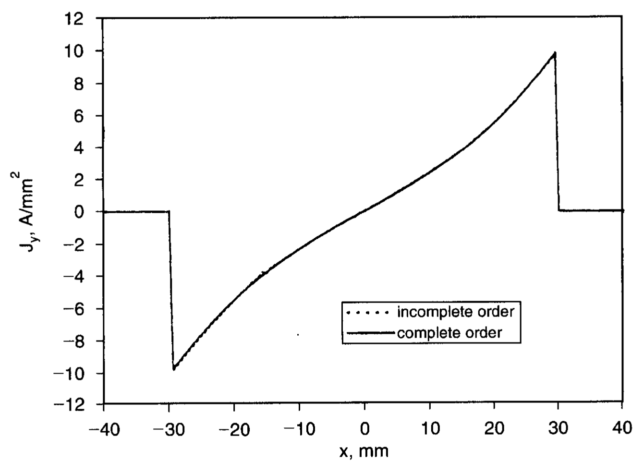

Fig. 4 *Variation of current density (imaginary part, 50 Hz, coarse mesh)*

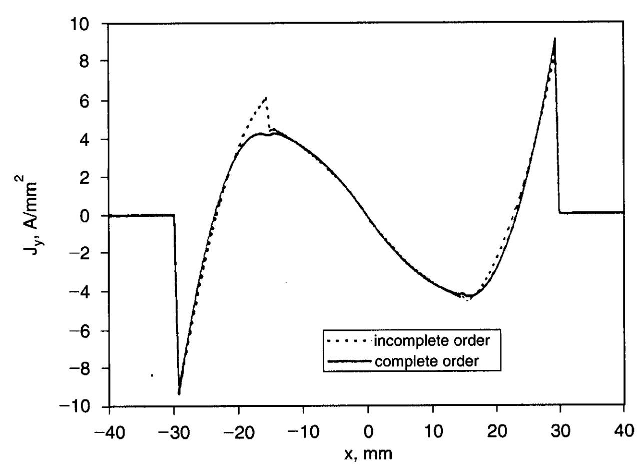

Fig. 5 *Variation of current density (imaginary part, 200 Hz, coarse mesh)*

The comparison is carried out using the same mesh and under two excitation frequencies: 50 Hz and 200 Hz. The skin depth (δ)/mesh size (h) ratios are, respectively, $\delta/h \approx 2.2$ at 50 Hz; and $\delta/h \approx 1.1$ at 200 Hz.

Figs. 4 and 5 plot the variation of the current density (y component, imaginary part) along the centre line ($y=0$, $z=0$) of the conductor at 50 Hz and 200 Hz. Results show that, at 50 Hz ($\delta/h \approx 2.2$), both the complete and incomplete bases give the same results. At 200 Hz, the skin depth is too small to accurately model the variation of the field with the incomplete basis. The accuracy depends strongly on the ratio δ/h. As long as the skin depth is much larger than the mesh size, we get the same accuracy from the elements of complete and incomplete order.

To confirm this point, we used a finer mesh with 3226 elements. This mesh contains 21865 unknowns for the incomplete basis and 32003 unknowns for the complete basis. The ratio δ/h is about 3.5 at 50 Hz and 1.8 at 200 Hz. Results at 50 Hz are almost the same as those obtained with the coarse mesh. The variation of the current density at 200 Hz is plotted in Fig. 6. We observe that the result for the incomplete-order elements becomes very close to that from the complete-order elements. If we increase the frequency to 500 Hz, the ratio δ/h diminishes to 1.1, and we observe again a difference between the results of complete and incomplete-order elements (Fig. 7). It has to be stated that when the skin depth/mesh size ratio becomes too small (<1), even the complete 2nd order element cannot model correctly the sharp variations of the field. The order of approximation has to be increased appropriately.

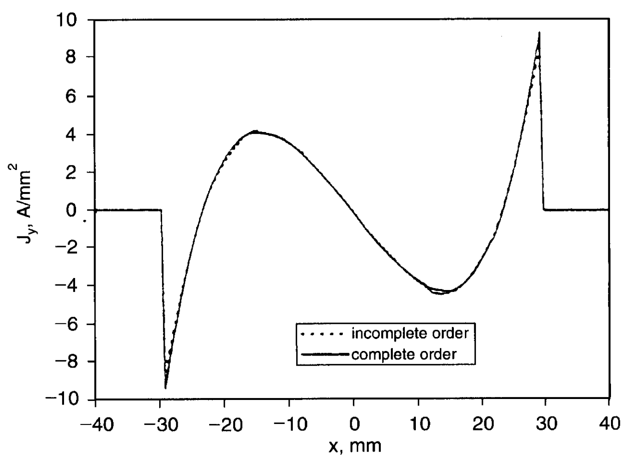

Fig. 6 *Variation of current density (imaginary part, 200 Hz, fine mesh)*

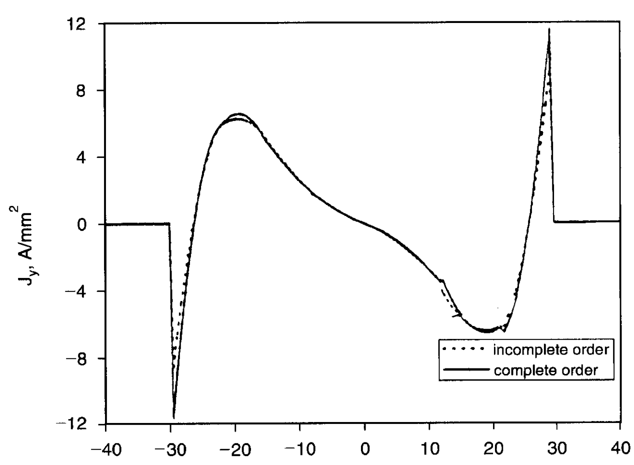

Fig. 7 *Variation of current density (imaginary part, 500 Hz, fine mesh)*

Table 6 compares the computation performance related to the elements of incomplete and complete bases. The number of iterations corresponds to the case when the relative error of the iterative solver converges to 10^{-9}.

Table 6: Comparison of the performance of elements of complete and incomplete bases (at 200 Hz)

Basis	No. of elements	No. of unknowns	Memory size Mb	No. of iterations	CPU time s
Incomplete	1600	10438	7.1	199	51
	3226	21865	12.6	198	112
Complete	1600	15414	14.9	214	119
	3226	32003	27.4	216	252

We conclude that the accuracy of the results obtained from the elements of complete and incomplete bases is of the same order once the ratio skin depth/mesh size is larger than 2. However, the memory size and CPU time are much larger with the complete-order element. This shows that the element of complete order is useful only in the case of small skin depth. The price to be paid is an increase in the computational effort.

5 Conclusions

The function spaces of q-order p-form elements of complete bases are analysed and compared with those of incomplete bases. The two kinds of bases differ only by a null space of the differential operator. They model the same range space. The use of a 1-form (res. 2-form) complete-order element for the modelling of a rotational field (res. divergent field) is redundant. In the case of eddy-current (or high-frequency) problems, the elements of complete order may allow a better approximation of the irrotational field. However, through the study of 2nd order 1-form hierarchical elements of complete and incomplete bases for the computation of eddy currents, we found that once the mesh size is fine enough (with the skin depth/mesh size ratio larger than 2), the element of incomplete order can provide good accuracy with much less computational effort. This conclusion may also be true for the case of high-frequency problems if the mesh size is small enough compared to the wavelength. Since, in most applications, the mesh fineness requirement is satisfied, the incomplete q-order element can provide good accuracy, is more economical and hence preferable.

6 References

1 NÉDÉLEC, J.C.: 'Mixed finite element in R^3', *Numer. Math.*, 1980, **35**, pp. 315–341

2 BOSSAVIT, A.: 'Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism', *IEE Proc. A., Phys. Sci. Meas. Inform. Manage. Elec. Rev.*, 1988, **135**, (8), pp. 493–500

3 MUR, G., and DE HOOP, A.T.: 'A finite element method for computing three dimensional electromagnetic field in inhomogeneous media', *IEEE Trans. Magn.*, 1985, **21**, (6), pp. 2188–2191

4 NÉDÉLEC, J.C.: 'A new family of mixed finite elements in R^3', *Numer. Math.*, 1986, **50**, pp. 57–81

5 REN, Z., and IDA, N.: 'High order differential form based elements for the computation of electromagnetic field', *IEEE Trans. Magn.*, 2000, **36**, (4), pp. 1472–1478

6 REN, Z., and IDA, N.: 'Solving 3D eddy current problems using second order nodal and edge elements', *IEEE Trans. Magn.*, 2000, **36**, (4)