Curvilinear and Higher Order 'Edge' Finite Elements in Electromagnetic Field Computation

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Abstract: - "Edge" type finite elements are very useful in computation of electromagnetic fields. Unlike nodal-based finite elements, they guarantee continuity of tangential components of the field variables across element interfaces while allowing discontinuity in normal components. This, in turn, eliminates the so-called spurious solutions in eigenvalue analysis in cavities. However, most of the work in this important area is done with linear, tetrahedral elements. A method for the systematic construction of first and second order "edge" and "facet" finite elements based on the nodal-based conventional elements, and their use is presented in this work. Higher order elements are also considered. Both tetrahedral and hexahedral elements are presented.

These elements are intimately related to the corresponding nodal-based elements, allowing an easy implementation in existing nodal-based finite element computer programs. The elements constructed are then used for mode analyses in electromagnetic cavities. Better solutions are obtained compared to linear elements.

I. INTRODUCTION

"Edge" and "facet" elements were originally proposed in [1] under the name "mixed finite elements". It was found later that Whitney's form provides good choices for "edge" and "facet" shape functions built on a linear tetrahedron [2]. In Cartesian coordinates, the tetrahedral "edge" shape functions have the simple form $\mathbf{a}+\mathbf{b}\times\mathbf{r}$. These shape functions are, in addition, divergence free [3]. A consistently linear tetrahedral "edge" element was provided in [4]. Linear hexahedral "edge" and "facet" elements were proposed in [5]. "Edge" finite elements are very attractive for electromagnetic field computation. Unlike nodal-based finite elements, they guarantee continuity of tangential components of the field variables across element interfaces while allowing discontinuity in normal components. However, to the author's knowledge, there is no systematic way to construct "edge" and "facet" elements comparable to the well established, conventional node-based finite elements, although methods of constructing "edge" finite elements exist [7,8]. This is explained further by the following observations. i) An "edge" type element, in contrast to a nodal element, has shape functions with both magnitudes and directions. The currently used linear tetrahedral element [2,3] and linear hexahedral element [5], can only provide a linear variation for directions. In order to have a second order variation in directions, a curvilinear mapping must be used. ii) The forementioned elements permit a linear interpolation for the field inside the element, but only a constant variation for the tangential component along the edge (or the normal component on a facet) except the element used in [4]. Obviously, to obtain a consistent linear interpolation, each edge must have *two* degrees of freedom [4]. Accordingly, each edge and each facet must have *three* degrees of freedom to permit a consistent second order interpolation of the field everywhere in the element, and so on.

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This paper presents a systematic method of construction of curvilinear and higher order "edge" and "facet" elements. The fundamental procedure, based on the work of [4] and [5], is demonstrated using curvilinear tetrahedral and hexahedral shapes. Curvilinear "edge" elements are then used for mode analyses in electromagnetic cavities; no spurious modes are observed.

II. CONSTRUCTION OF CURVILINEAR ELEMENTS

Our objective here is to construct vector shape functions $\mathbf{w}_i(\mathbf{r})$ which can be used to describe vector functions of 1-form and 2-form. One of the basic requirement is that $\mathbf{w}_i(\mathbf{r})$ should guarantee tangential continuity in edge elements and normal continuity in facet elements. These vector shape functions are easier to construct in local coordinates, as for the nodal elements. Let the vector shape functions have the following form:

$$\mathbf{w}_i(\mathbf{r}) = \phi_i(\xi,\eta,\zeta)\mathbf{v}_i(\mathbf{r}), \quad i=1, \ldots\ldots, M \tag{1}$$

where $\phi_i(\xi,\eta,\zeta)$ are completely defined in the local system of coordinates. Since direction vectors are defined in the global coordinates, no misunderstanding will be introduced. The joint effect of ϕ_i and $\mathbf{v}_i$ ensures that $\mathbf{w}_i$ has a unit projection at node i, and zero projection at all other nodes. The following is a systematic procedure of constructing higher order and curvilinear vector shape functions:

1. The geometric factor is included in $\mathbf{v}_i$ and ϕ_i decides the order of the element;
2. use a standard nodal shape function in local coordinates as ϕ_i. Normalize ϕ_i such that it is equal to 1 at node i and zero at other nodes;
3. choose $\mathbf{v}_i$ in an edge element along normal directions of parametric planes;
4. choose $\mathbf{v}_i$ in a facet element along parametric lines;
5. normalize $\mathbf{v}_i$ by the Jacobian [4] such that $\mathbf{v}_i$ in an edge element has unit tangential projection along the edge direction; vi in a "facet" element has a unit normal projection along the facet normal.

Next, we use this basic procedure to build vector shape functions for two commonly used shapes.

A. Hexahedral Elements

Consider first a hexahedral element in Cartesian coordinates (x,y,z) as shown in Fig. 1.It can be mapped into a unit box in the local coordinates (ξ,η,ζ), through the following coordinate transformation:

$$\mathbf{r} = \sum_{j=1}^{n} N_j(\xi,\eta,\zeta)\mathbf{r}_j \tag{2}$$

where N_i are the shape functions of a nodal element, N is the number of nodes in the element, and $\mathbf{r}_i$ are the corresponding position vectors. The three vectors along the parametric lines in the three directions are

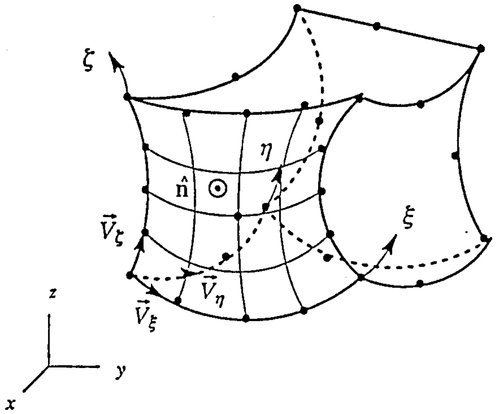

Fig. 1. A Curvilinear Hexahedron

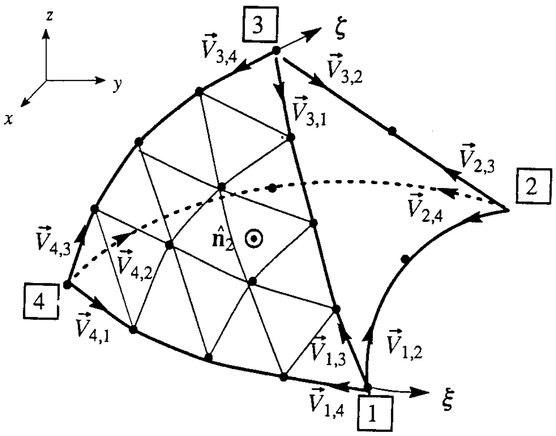

Fig. 2. A Curvilinear Tetrahedron

$$\mathbf{V}_\xi = \frac{\partial \mathbf{r}}{\partial \xi}, \quad \mathbf{V}_\eta = \frac{\partial \mathbf{r}}{\partial \eta}, \quad \mathbf{V}_\zeta = \frac{\partial \mathbf{r}}{\partial \zeta} \tag{3}$$

Accordingly, the three vectors along the three parametric planes are $\mathbf{V}_\eta \times \mathbf{V}_\zeta$, $\mathbf{V}_\zeta \times \mathbf{V}_\xi$, $\mathbf{V}_\xi \times \mathbf{V}_\eta$. These vectors can be chosen as $\mathbf{v}_i$. For example, for "edge" elements, $\mathbf{v}_i$ are chosen as [4]

$$\mathbf{v}_i = \frac{\mathbf{V}_\eta \times \mathbf{V}_\zeta}{\mathbf{V}_\xi \cdot \mathbf{V}_\eta \times \mathbf{V}_\zeta} \tag{4}$$

where the normalizing factor is the Jacobian. Obviously, (4) has a continuous tangential component across elemental interfaces. Similarly, for "facet" shape functions for facets parallel to the $\eta-\zeta$ plane, $\mathbf{v}_i$ are chosen as

$$\mathbf{v}_i = \frac{\mathbf{V}_\xi \left| \mathbf{V}_\eta \times \mathbf{V}_\zeta \right|}{\mathbf{V}_\xi \cdot \mathbf{V}_\eta \times \mathbf{V}_\zeta} \tag{5}$$

(5) has a continuous normal component across elemental interfaces.

B. Tetrahedral Elements

Consider now a curvilinear tetrahedral element in the Cartesian coordinates (x,y,z) as shown in Fig. 2. As usual, volume coordinates (L_1,L_2,L_3,L_4) are introduced and the following representation is obtained:

$$\mathbf{r} = \sum_{j=1}^{N} N_j(L_1,L_2,L_3,L_4)\mathbf{r}_j \tag{6}$$

For each corner node $k=1,2,3,4,$ three parametric directions $i=1,2,3,4,\ i\neq k,$ can be defined:

$$\mathbf{V}_{k,i} = \sum_{j=1}^{N} \left(\frac{\partial N_j}{\partial L_i} - \frac{\partial N_j}{\partial L_k} \right) \mathbf{r}_j \tag{7}$$

by using $L_1+L_2+L_3+L_4=1$ in equation (6). We observe that $\mathbf{V}_{k,i}=-\mathbf{V}_{i,k}$. Consider edge $e_{4,1}$. Assume that the direction of edge $e_{4,1}$ is from $\mathbf{r}_4$ to $\mathbf{r}_1$. The two families of parametric planes intersecting this edge have normal directions along $\mathbf{V}_{4,2}\times\mathbf{V}_{4,3}$ and $\mathbf{V}_{1,3}\times\mathbf{V}_{1,2}$ respectively. To maintain symmetry, the direction vector for any edge shape function on this edge is chosen as a linear interpolation of these two normal directions:

$$\mathbf{v}_i = \left[(1-\xi_i) \frac{\mathbf{V}_{4,2}\times\mathbf{V}_{4,3}}{\mathbf{V}_{4,1}\cdot\mathbf{V}_{4,2}\times\mathbf{V}_{4,3}} - \xi_i \frac{\mathbf{V}_{1,3}\times\mathbf{V}_{1,2}}{\mathbf{V}_{1,4}\cdot\mathbf{V}_{1,3}\times\mathbf{V}_{1,2}} \right] \tag{8}$$

where $\xi_i=0$ for node 4, 1 for node 1, $1/2$ for the central node, etc. Obviously, $\mathbf{v}_i$ has a unit tangential projection along the vector $\mathbf{V}_{4,1}$. For "facet" shape functions, take facet f_4 (formed by three corner nodes $1,2,3$) for example. Let the area coordinates for any node i on facet f_4 be (ξ_i,η_i,ζ_i). The following vector is chosen as the "facet" shape direction vector $\mathbf{v}_i$:

$$\mathbf{v}_i = \left[\xi_i \frac{\mathbf{V}_{4,1}\left|\mathbf{V}_{1,2}\times\mathbf{V}_{1,3}\right|}{\mathbf{V}_{4,1}\cdot\mathbf{V}_{1,3}\times\mathbf{V}_{1,4}} + \eta_i \frac{\mathbf{V}_{4,2}\left|\mathbf{V}_{2,3}\times\mathbf{V}_{2,1}\right|}{\mathbf{V}_{4,2}\cdot\mathbf{V}_{2,3}\times\mathbf{V}_{2,1}} + \zeta_i \frac{\mathbf{V}_{4,3}\left|\mathbf{V}_{3,1}\times\mathbf{V}_{3,2}\right|}{\mathbf{V}_{4,3}\cdot\mathbf{V}_{3,1}\times\mathbf{V}_{3,2}} \right] \tag{9}$$

III. EIGENMODE ANALYSES IN CAVITIES

The finite element method provides a powerful tool for finding modes (especially the lower modes) in arbitrarily shaped, inhomogeneously loaded electromagnetic cavities. Unfortunately, the standard finite element solution to the vector eigenvalue problem in a cavity consists of both physical and nonphysical (or spurious) modes. Since spurious solutions have a relatively higher divergence, a number of methods have been established to eliminate or reduce nonphysical modes by enforcing the divergence free condition. Among these, the penalty method and the reduction method are commonly used [6]. In this section, we use "edge" elements to solve the problem.

A. Formulation

The problem under consideration is to find the eigenmodes of a dielectric loaded EM cavity. The cavity wall, S, is assumed to be of arbitrary shape. The interior of the cavity, Ω, is characterized by $(\mu_0\mu_r, \varepsilon_0\varepsilon_r(\mathbf{r}))$, where μ_0 and ε_0 are free space permeability and permittivity, respectively. Consider the $\mathbf{E}$ formulation only. With $e^{j\omega t}$ variation implied, the modes in the cavity satisfy the following weak form:

$$\int_\Omega \left(\frac{1}{\mu_r}\nabla\times\mathbf{E}\right)\cdot(\nabla\times\mathbf{w}_m)d\Omega - k_0^2\int_\Omega \varepsilon_r(\mathbf{r})\mathbf{E}\cdot\mathbf{w}_m d\Omega = jk_0\eta_0\int_s \left(\hat{\mathbf{n}}\times\mathbf{H}\right)\cdot\mathbf{w}_m dS \qquad (10)$$

where $\mathbf{w}_m$ is any set of real vector weighting functions, and k_0 is the free-space wavenumber. k_0 $(rads/m)$ is treated as the resonant frequency for numerical convenience. By virtue of the weak form, we can define a homogeneous boundary condition $\hat{\mathbf{n}}\times\mathbf{E}$ (electric wall) and a natural boundary condition $\hat{\mathbf{n}}\times\mathbf{H}$. By introducing the following expansion:

$$\mathbf{E}(\mathbf{r}) = \sum_{n=1}^{N} E_n\mathbf{w}_n(\mathbf{r}) \qquad (11)$$

where $\mathbf{w}_n$ are the proposed "edge" element shape functions, E_n are the tangential projections of $\mathbf{E}$ along edge directions, and N is the total number of unknowns. Since $\mathbf{w}_n$ guarantee tangential continuities of $\mathbf{E}$ on the elemental interface, no special treatment is needed on material interfaces. With the vector weighting functions chosen to be the same as the shape functions $\mathbf{w}_n$, (10) is reduced to a real symmetric eigensystem: $[A]\{E\} = k_0^2[B]\{E\}$. The system is solved by Lanczos' method.

B. Examples

Curvilinear elements provide a better description to curved cavity walls than linear elements. As a result, a relatively small mesh can yield good predictions to the lower resonant frequencies. A comparison was made between quasi-linear hexahedral models and quasi-curvilinear hexahedral models (12 unknowns per element) for an empty spherical cavity. As shown in Fig. 3, only one-eighth of the sphere is needed. Table I shows the dominant mode computed by the present model using different mesh sizes. The advantage of curvilinear elements is clearly demonstrated.

Table I. Dominant Mode in an Empty Spherical Cavity

Element	Unknowns	Numerical	Analytical	Error
8 node	96	2.8541	2.744	+4.0%
(Linear)	324	2.8005	2.744	+2.1%
	768	2.7818	2.744	+1.3%
20 node	96	2.7862	2.744	+1.5%
(Curvi-	324	2.7709	2.744	+1.0%
linear)	768	2.7653	2.744	+0.8%

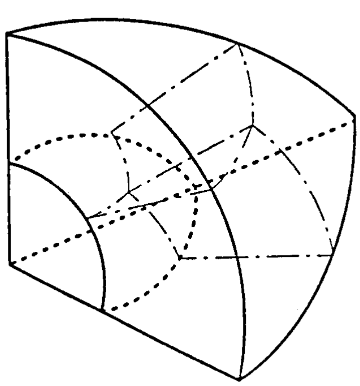

Fig. 3. Modeling of a Spherical Cavity

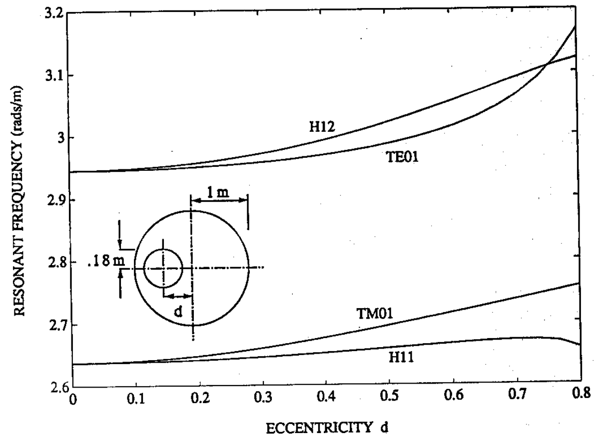

Fig. 4. Four Lower Modes of a Spherical Cavity Containing an Eccentric Sphere.

The next example is a dielectric sphere (radius=0.18m) with ε_r=36 eccentrically loaded in a spherical cavity (radius=1m). Two way symmetry is employed. Fig. 4 shows the four lower modes when the eccentricity is changed.

IV. CONCLUSIONS

Curvilinear and higher order hexahedral and tetrahedral "edge" and "facet" elements were constructed. These elements are rather general, in the sense that most elements currently used are simplified versions of the presently proposed elements. These vector elements are intimately related to the corresponding nodal elements, allowing simple implementation in existing nodal-based finite element computer programs. Some curvilinear "edge" elements were used to compute eigenmodes of cavities, with improved accuracies over linear elements; no spurious modes were present. Vector boundary elements were also developed but not included in the present paper. These elements are currently used in the finite element and moment method analysis of electromagnetic scattering problems.

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