Selection of the Surface Impedance Boundary Conditions for a Given Problem

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Abstract - **It is well-known that the surface impedance boundary conditions (SIBCs) are classified by their order of approximation. In the present paper *universal* relationships between characteristic values of the problem are proposed considering the order of approximation so that the SIBCs, best suited to a given problem, can be easily selected. The methodology is applicable to the time- frequency- domain linear and non-linear SIBCs. Numerical examples are included to illustrate the methodology.**

Index terms - **low penetration problems, asymptotic expansions analysis, approximate boundary conditions, surface impedance boundary condition, skin effect, perturbation methods.**

I. INTRODUCTION

Calculation of the field distribution in the conductor and surrounding space under condition of the skin effect is a classical problem in engineering electromagnetics. If the main interest is focused on the field in an exterior region, use of the skin effect theory can essentially simplify the problem, namely: the conducting volume can be replaced by the approximate boundary conditions (ABCs) applied at its boundary and eliminated from the numerical procedure. These ABCs take into account the material properties of the conductor and provide approximate relationships between the tangential components of the electric and magnetic fields or between the normal and tangential components of the magnetic field. By analogy with circuit theory where the ratio between the voltage and current has been denoted by the term "impedance", the ABCs received the name surface impedance boundary conditions (SIBCs).

The surface impedance concept was introduced in early 1940's [1-3]. The basic concept is the following. If the skin depth in the conducting body is so short that the variation of the field in the direction tangential to the body's surface is much less than the field variation in the normal direction, then the original 3-D equation of the electromagnetic field diffusion into the body can be replaced by a 1-D equation in the direction normal to the surface of the body. Analytical (in the linear case) or numerical (in the non-linear case) solution of the reduced equation can be then used to derive the SIBCs on the body's surface.

The original frequency domain conditions have been transformed to the time domain form [4-6]. The smooth surface SIBCs have been extended to corners [7-9]. Recently, SIBCs for non-linear problems have been developed [10-11]. At present various SIBCs of different approximation order are used in combination with the BE, FE and FDTD methods for analysis of a wide range of practical applications such as transformers, inductive heating devices, microstrip lines, HF power applications, transmission lines, plasma and magnetic levitation devices, non-destructive testing analysis, electromagnetic scattering, geophysical problems, etc. The correct choice of the SIBCs for a given problem is very important and not always clear *a priori*, especially in the transient case. If the order of the approximation of the SIBCs used in the computation is inadequate, then the computational results are not validated enough. On the other hand, application of the high order SIBCs for the calculation of very thin skin layers does not provide gain in accuracy of the results and leads to useless computational expenses. In this paper we propose the methodology how to select the SIBCs, that are best suited for a given problem, using the characteristic values of this problem.

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II. CLASSIFICATION OF SIBCS BY APPROXIMATION ORDER

The SIBCs developed by now for modelling of homogeneous isotropic conducting bodies can be classified by the order of approximation into the following classes.

1. The Leontovich approximation, in which the body's surface is considered as a plane and the field is assumed to be penetrating into the body only in the direction normal to the body's surface. The SIBCs in the Leontovich approximation can be written in the frequency and time domain as follows:

$$\dot{E}_{\vartheta_k} = (-1)^{3-k}\frac{1+j}{2\theta}\mu\omega\delta\dot{H}_{\vartheta_{3-k}}, \qquad k=1,2 \qquad (1a)$$

$$\delta = \sqrt{2/(\omega\sigma\mu)}; \qquad \theta = \sqrt{1+j\varepsilon\omega/\sigma}$$

$$E_{\vartheta_k} = (-1)^{3-k}T_1 * H_{\vartheta_{3-k}} \qquad (1b)$$

$$T_1(t) = \left(\frac{\mu}{\varepsilon}\right)^{1/2}\left\{U(t)+\frac{\sigma}{2\varepsilon}\left[I_1\left(\frac{\sigma t}{2\varepsilon}\right)-I_0\left(\frac{\sigma t}{2\varepsilon}\right)\right]\exp\left(-\frac{\sigma t}{2\varepsilon}\right)\right\}$$

$$T_1(t)\big|_{\varepsilon/\sigma t<<1} = -\sqrt{\mu/2\pi\sigma t^3}$$

Here ω is the angular frequency of the field source, σ, μ and ε are, respectively, the electrical conductivity, magnetic permeability and dielectric permittivity of the body, "*" denotes time-convolution product, $I_n(x)$ is the modified Bessel function of order n, $U(t)$ is the unit step function and ϑ_k, $k=1,2$, are the principal curvature coordinates defined as

$$\vec{e}_{\vartheta_1}\times\vec{e}_{\vartheta_2} = \vec{e}_\eta = \vec{n}$$

where $\vec{e}_{\vartheta_1}, \vec{e}_{\vartheta_2}, \vec{e}_\eta$ are the basis unit vectors of the system and the unit normal vector $\vec{n}$ is directed inside the body.

2. The Mitzner approximation, in which the curvature of the body's surface is taken into account, but the diffusion is assumed to be only in the direction normal to the surface as in the Leontovich approximation. The frequency and time domain SIBCs in the Mitzner approximation can be written in the form:

$$\dot{E}_{\vartheta_k} = (-1)^{3-k}\frac{1+j}{2\theta}\mu\omega\delta\left[1+\frac{1-j}{4\theta}\delta\left(d_{3-k}^{-1}-d_k^{-1}\right)\right]\dot{H}_{\vartheta_{3-k}} \quad (2a)$$

$$E_{\vartheta_k} = (-1)^{3-k}\left[T_1+\left(d_{3-k}^{-1}-d_k^{-1}\right)T_2\right] * H_{\vartheta_{3-k}} \quad (2b)$$

$$T_2(t)=\varepsilon^{-1}\exp(-\sigma t/\varepsilon); \qquad T_2(t)\big|_{\varepsilon/\sigma t\ll 1} = U'(t)/\sigma$$

where d_k, $k=1,2$, are the local radii of curvature of the coordinate line ϑ_k and $U'(t)$ is the Dirac function.

3. The Rytov approximation, in which the field diffusion in the directions tangential to the body's surface is taken into account. The frequency and time domain SIBCs in the Rytov approximation can be written in the form:

$$\dot{E}^s_{\vartheta_k} = (-1)^{3-k}\frac{1+j}{2\theta}\mu\omega\delta\left\{\left[1+\frac{1-j}{4\theta}\delta\left(d_{3-k}^{-1}-d_k^{-1}\right)\right]\dot{H}^s_{\vartheta_{3-k}} + \right.$$
$$+\frac{\delta^2}{2j\theta}\frac{3d_k^2-d_{3-k}^2-2d_kd_{3-k}}{8d_k^2d_{3-k}^2}\dot{H}^s_{\vartheta_{3-k}} +$$
$$\left.+\frac{\delta^2}{4j\theta}\left(-\frac{\partial^2\dot{H}_{\vartheta_{3-k}}}{\partial\vartheta_k^2}+\frac{\partial^2\dot{H}_{\vartheta_{3-k}}}{\partial\vartheta_{3-k}^2}+2\frac{\partial^2\dot{H}_{\vartheta_k}}{\partial\vartheta_k\partial\vartheta_{3-k}}\right)\right\} \quad (3a)$$

$$E_{\vartheta_k} = (-1)^{3-k}\left\{T_1 * H_{\vartheta_{3-k}} + \left(d_{3-k}^{-1}-d_k^{-1}\right)T_2 * H_{\vartheta_{3-k}} + \right.$$
$$+\frac{3d_k^2-d_{3-k}^2-2d_kd_{3-k}}{8d_k^2d_{3-k}^2}T_3 * H_{\vartheta_{3-k}} +$$
$$\left.+\frac{1}{2}T_3 * \left(-\frac{\partial^2 H_{\vartheta_{3-k}}}{\partial\vartheta_k^2}+\frac{\partial^2 H_{\vartheta_{3-k}}}{\partial\vartheta_{3-k}^2}+2\frac{\partial^2 H_{\vartheta_k}}{\partial\vartheta_k\partial\vartheta_{3-k}}\right)\right\} \quad (3b)$$

$$T_3(t) = t\left(\varepsilon\sqrt{\varepsilon\mu}\right)^{-1}\left[I_0(\sigma t/(2\varepsilon)) - I_1(\sigma t/(2\varepsilon))\right]\exp(-\sigma t/(2\varepsilon))$$

$$T_3(t)\big|_{\varepsilon/\sigma t\ll 1} = \left(\pi\sigma^3\mu t\right)^{-1/2}$$

The SIBCs of the first class are called low-order conditions while the SIBCs of the second and third classes that are high-order conditions. As can be noted from (1)-(3), a SIBC of lower order is included in a SIBC of higher order. Therefore, use of the low-order SIBC in a given problem means that the high-order terms are neglected.

II. Characteristic Values of the Problem

Any non-static electromagnetic problem involves the following scales: characteristic dimension D^* of the body's surface and characteristic time τ^*. We use these values as "input data" in our methodology.

Characteristic time is defined as the ratio $2/\omega$ for time-harmonic incident field and the incident pulse duration τ_p for the pulsed source.

Characteristic dimension D^* is defined as

$$D^* = \min\left(R_\vartheta, R_s\right)$$

where R_ϑ is minimum radius of the curvature of the surface coordinate lines in the case of smooth body and R_s is the minimum distance between the field source and the body.

Using D^* and τ^*, we define the characteristic skin depth δ and characteristic dimension λ of the field variation along the body's surface as follows:

$$\delta = \sqrt{\tau^*/(\mu\sigma)} \quad (4)$$

$$\lambda = c\tau^* \quad (5)$$

where c is the velocity of light.

If the material properties are non-linear, the characteristic permeability μ^* (reluctivity ν^*) and conductivity σ^* should be used in (4). Since the electrical conductivity-temperature relationship and BH-curve of the conductors are assumed to be known, values ν^* and σ^* can be found as follows

$$\nu^* = \nu^*(H^*); \qquad \sigma^* = \sigma^*(T^*);$$

where the characteristic magnetic field H^* and temperature T^* can be expressed in the terms of D^*, τ^* and characterstic current I^* as

$$H^* = I^*/\left(4\pi D^*\right); \qquad T^* = I^{*2}/\left(4\pi c\rho\nu^* D^{*2}\right)$$

More details on the choice of the characteristic values in non-linear case can be found in [12].

The conditions of applicability of the surface impedance concept can be easily written in terms of D^*, δ and λ:

$$\delta \ll D^* \quad \text{or} \quad p = \delta/D^* \ll 1 \quad (6)$$

$$\lambda \gg D^* \quad \text{or} \quad q = D^*/c\tau^* \ll 1 \quad (7)$$

where p and q are basic parameters of the problem. The conditions (6)-(7) hold for ***all*** SIBCs.

III. SIBCs as Asymptotic Expansions

As has been shown by Rytov [1], an SIBC can be represented in the form of asymptotic expansions in the small parameter equal to the ratio of the characteristic penetration depth and characteristic dimension of the body's surface:

$$\tilde{E}_{\vartheta_k} = (-1)^k \sum_{k=0}^{\infty} p^{k+1} Z_k\left(\tilde{H}_{\vartheta_{3-k}}\right); \quad (8)$$

where "~" denotes the non-dimensional values obtained by introducing the following scale factors [12]:

$$[\vartheta_1, \vartheta_2] = D^*; \quad [\eta] = pD^*; \quad [H] = \frac{I^*}{4\pi D^*}; \quad [E] = \frac{\mu I^*}{4\pi\tau^*}$$

Here the square brackets denote a scale factor for the corresponding value.

The representation (8) has a clear physical meaning, namely: the zero-order terms ($p=0$) in the expansions are the conditions on the surface of perfect electrical conductor (so-called PEC-limit, in which the field diffusion into the body is neglected). The first-order, second-order and third-order terms give the corrections of the order of the Leontovich, Mitzner and Rytov approximations, respectively.

The SIBCs (3a)-(3b) are represented in the form (8) as follows:

$$\dot{\tilde{E}}_{\vartheta_k} = (-1)^{3-k} p \frac{1+j}{\theta} \left\{ \dot{\tilde{H}}_{\xi_{3-k}} + p \frac{1-j}{4\theta} \left(\tilde{d}_{3-k}^{-1} - \tilde{d}_k^{-1} \right) \dot{\tilde{H}}_{\xi_{3-k}} + \right.$$
$$+ \frac{p^2}{2j\theta} \frac{3\tilde{d}_k^2 - \tilde{d}_{3-k}^2 - 2\tilde{d}_k \tilde{d}_{3-k}}{8\tilde{d}_k^2 \tilde{d}_{3-k}^2} \dot{\tilde{H}}_{\vartheta_{3-k}} +$$
$$\left. + \frac{p^2}{2j\theta} \left(- \frac{\partial^2 \dot{H}_{\vartheta_{3-k}}}{\partial \vartheta_k^2} + \frac{\partial^2 \dot{H}_{\vartheta_{3-k}}}{\partial \vartheta_{3-k}^2} + 2 \frac{\partial^2 \dot{H}_{\vartheta_k}}{\partial \vartheta_k \partial \vartheta_{3-k}} \right) \right\} + O(p^4) \quad (9a)$$

$$\tilde{E}_{\vartheta_k} = (-1)^k p \left\{ \tilde{T}_1 * \tilde{H}_{\vartheta_{3-k}} + p \left(\tilde{d}_{3-k}^{-1} - \tilde{d}_k^{-1} \right) \tilde{T}_2 * \tilde{H}_{\vartheta_{3-k}} + \right.$$
$$+ p^2 \frac{3\tilde{d}_k^2 - \tilde{d}_{3-k}^2 - 2\tilde{d}_k \tilde{d}_{3-k}}{8\tilde{d}_k^2 \tilde{d}_{3-k}^2} \tilde{T}_3 * \tilde{H}_{\vartheta_{3-k}} +$$
$$\left. + \frac{p^2 \tilde{T}_3}{2} * \left(- \frac{\partial^2 \tilde{H}_{\vartheta_{3-k}}}{\partial \tilde{\vartheta}_k^2} + \frac{\partial^2 \tilde{H}_{\vartheta_{3-k}}}{\partial \tilde{\vartheta}_{3-k}^2} + 2 \frac{\partial^2 \tilde{H}_{\vartheta_k}}{\partial \tilde{\vartheta}_k \partial \tilde{\vartheta}_{3-k}} \right) \right\} + O(p^4) \quad (9b)$$

Because in the expansions (9) the terms of the order $O(p^4)$ have been neglected, condition (6) can be replaced by:

$$p^4 \ll 1 \quad (10)$$

IV. METHODOLOGY

Now we can evaluate the ranges of the characteristic values for which the SIBC (9) (and, consequently, (3)) are best applicable. The conditions (6)-(7) of applicability of the SIBCs involve two characteristic scales (D^* and τ^*) and two parameters of the problem (p and q), therefore, the scales are *uniquely* expressed by these parameters.

From (10) it follows that approximation errors of the PEC-limit, the Leontovich SIBCs, the Mitzner SIBCs and the Rytov SIBCs are p, p^2, p^3 and p^4, respectively. Then we can define the approximate range of the parameter p, for which the SIBCs of these classes can be best applied:

1. The PEC boundary conditions
$$p < 0.06 \quad (11a)$$
2. The SIBCs in the Leontovich approximation
$$p = 0.06 \div 0.25 \qquad (p^2 \approx 0.003 \div 0.06) \quad (11b)$$
3. The SIBCs in the Mitzner approximation
$$p = 0.25 \div 0.4 \qquad (p^3 \approx 0.02 \div 0.06) \quad (11c)$$
4. The SIBCs in the Rytov approximation
$$p = 0.4 \div 0.5 \qquad (p^4 \approx 0.03 \div 0.06) \quad (11d)$$

The range of the parameter q can be defined as
$$q < 0.06 \quad (12)$$

Under the definition (11)-(12) the approximation error due to using the SIBCs will not exceed 6%.

Introduce the following functions φ and ψ from (6)-(7):

$$D^* = p^{-1}(\sigma\mu)^{-1/2}(\tau^*)^{1/2} = \varphi(\tau^*, p) \quad (13)$$

$$D^* = cq\tau^* = \psi(\tau^*, q) \quad (14)$$

By substituting the extreme values (11)-(12) of the parameters p and q into the functions (13)-(14), the desired ranges of the scales D^* and τ^* can be obtained for a given problem.

Note that it would be preferable to eliminate the properties of the conducting material from the functions (13)-(14). For this purpose we introduce the following non-dimensional variables

$$\tilde{D}^* = \sigma\mu c D^* \quad (15)$$

$$\tilde{\tau}^* = \sigma\mu c^2 \tau^* \quad (16)$$

With the variables (15)-(16) the functions (13)-(14) can be written in the following form:

$$\tilde{D}^* = q\tilde{\tau}^* = \tilde{\varphi}(\tilde{\tau}^*, q) \quad (17)$$

$$\tilde{D}^* = p^{-1}(\tilde{\tau}^*)^{1/2} = \tilde{\psi}(\tilde{\tau}^*, p) \quad (18)$$

The distributions of the functions (17)-(18) for the extreme values of the parameters p and q given by (11)-(12) are shown in Fig. 1.

Regions (1a)-(1d) denote the application area of the SIBCs in the PEC-limit, in the Leontovich approximation, in the Mitzner approximation and in the Rytov approximation, respectively. If the point in the $\tilde{D}^*$-$\tilde{\tau}^*$ plane lies in the regions 2 and 3, then the surface impedance concept can not be applied because the conditions (11)-(12) break down.

Therefore, to select the SIBCs for a given problem knowing the characteristic values D^* and τ^* and the material properties σ and μ, we should

1. calculate non-dimensional values $\tilde{D}^*$ and $\tilde{\tau}^*$ using (15)-(16);
2. find the appropriate point in the $\tilde{D}^*$-$\tilde{\tau}^*$ plane;
3. if this point lies in the regions 1a-1d, choose corresponding SIBCs from (3a)-(3b) or the PEC boundary conditions.

As a conclusion of this section let us emphasize, that the functions shown in Fig. 1 are *universal* because they do not depend on the properties of the conductor material.

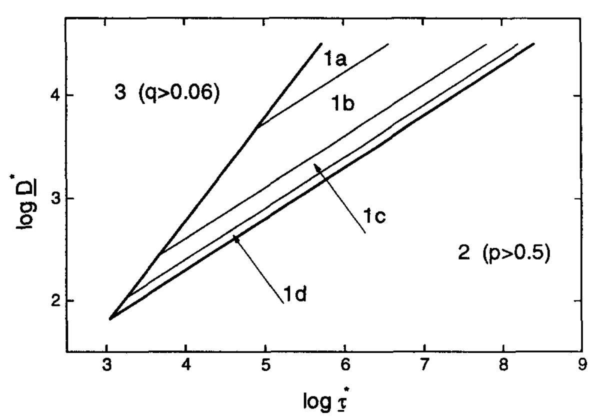

Fig. 1. $\tilde{D}^* - \tilde{\tau}^*$ plane

V. NUMERICAL EXAMPLE

The SIBCs (1b), (2b) and (3b) were coupled with the surface integral equation using the technique described in [13] and the formulation developed was solved by the boundary element method. We considered a pair of identical copper parallel conductors with circular cross section where equal and opposite directed single trapezoidal pulses of current 1 *A* are flowing from an external source as shown in Fig. 2. Radius of each conductor and the distance between the conductors were taken equal to 0.1 *m* (characteristic value D^*). Under these conditions the current density has only one component directed along the conductors.

To illustrate the theory, the distributions of the surface current density over one half of the cross section of one conductor were calculated for the following current pulses:

1. $\tau^* = 10^{-3}$ *s* ($p = 3.7 \cdot 10^{-2}$ and $q = 3.3 \cdot 10^{-7}$). From (11)-(12) it follows that the PEC-conditions are suitable for this problem. Figure 2 shows that the use of the SIBC of the next order (Leontovich's SIBC) will not provide significant increase of the accuracy of the results (difference between the curves does not exceed 4%).
2. $\tau^* = 10^{-2}$ *s* ($p = 1.2 \cdot 10^{-1}$ and $q = 3.3 \cdot 10^{-8}$). In this case the Leontovich SIBC seems to be optimal. From Figure 3 it follows that the difference between the curves obtained using the PEC- and Leontovich conditions is about 15% whereas application of the Mitzner SIBC increases the accuracy by 2% only.
3. $\tau^* = 10^{-1}$ *s* ($p = 3.7 \cdot 10^{-1}$ and $q = 3.3 \cdot 10^{-9}$). Figure 4 shows that the use of the Leontovich SIBC leads to the inpermissible computational error (about 18%). On the other hand, difference between the curves obtained in the Mitzner and Rytov approximations does not exceed 3%. Therefore, in this problem it is necessary to use the Mitzner SIBC as the methodology predicts.

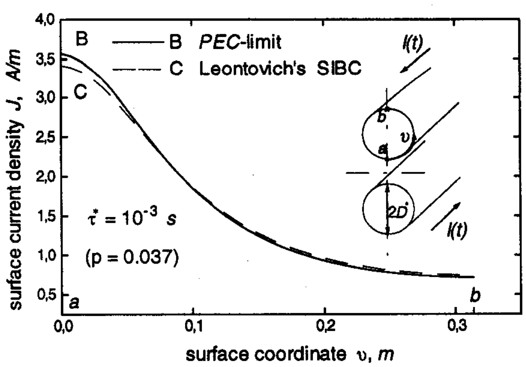

Fig. 2. Distribution of the surface current density over the conductor surface

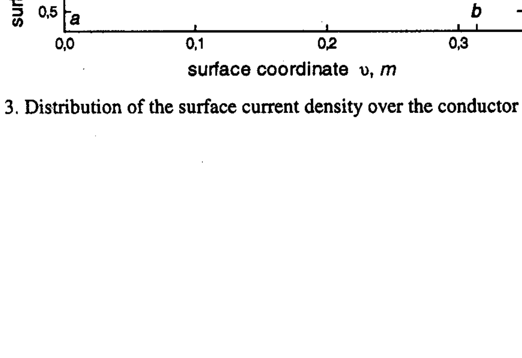

Fig. 3. Distribution of the surface current density over the conductor surface

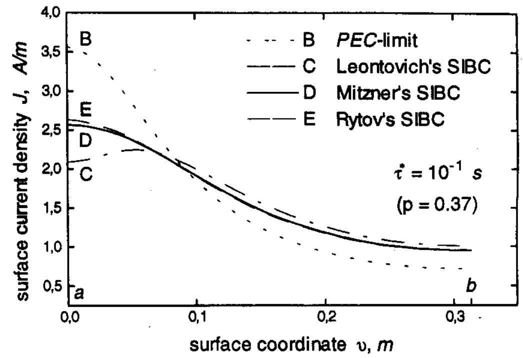

Fig. 4. Distribution of the surface current density over the conductor surface

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