Invariant BEM-SIBC Formulations for Time- and Frequency-Domain Eddy Current Problems

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***Abstract*—We demonstrate that frequency- and time-domain boundary integral equation formulations involving the surface impedance boundary conditions (SIBC's) can be transformed to invariant forms that depend only on the geometry of the problem and do not contain temporary parameters of the field. Thus the integral equations have to be solved only once for a given system of conductors and then the results for any source field (steady state or transient) can be easily obtained. We considered both low- and high-order SIBC's so that the formulations obtained can be applied to a wide range of skin effect problems. A numerical example is included to illustrate the theory.**

***Index Terms*—Boundary element method, perturbation methods, skin effect, surface impedance boundary condition, surface integral equations.**

I. INTRODUCTION

IN RECENT years the boundary element method (BEM) has come to be widely used for calculations of electromagnetic field under condition of skin effect in systems of conductors. Indeed, if the skin depth remains small, this method gives a practically ideal scheme for solving the problem: for functions in the dielectric region one writes an integral equation over the surface of the conductor with the use of the fundamental solution of Laplace's equation, and extra unknowns are eliminated by means of surface impedance boundary conditions (SIBC's) which can be of low- or high-order of approximation [1], [2]. However, the formulations constructed by this scheme have one general shortcoming: the integral equations contain parameters or operators specifying variation of the field in time (angular frequency for time-harmonic case or time-convolution integrals for transient case). Thus, if these parameters are changed, the integral equations must be solved anew, and for a transient field this must be done at each time step. Therefore, it is of theoretical and practical importance to derive an invariant form of the formulation of the problem in which the integral equations for a given system of conductors are determined solely by the geometrical parameters of the system and do not depend on the time dependence of the passage of current. The invariant forms for 2-D formulations in terms of E–H and A–V formalisms involving low order SIBC's were developed in [3]. In this paper we present a general technique for both low- and high-order SIBC's. The magnetic scalar potential formalism is used to cover 3-D problems.

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II. BOUNDARY INTEGRAL EQUATION IN TERMS OF MAGNETIC SCALAR POTENTIAL FORMALISM

Consider a system of N cylindrical conductors of homogeneous isotropic magnetic material in which currents

$$\vec{I}_i(\vec{r}, t) = \vec{I}_i^r(\vec{r})T_0(t), \qquad i = 1, \cdots, N, \tag{1}$$

flow from an external source. We use the following decomposition of the magnetic field in free space to introduce the magnetic scalar potential ϕ:

$$\vec{H} = \vec{H}_{fil} - \nabla\phi; \tag{2}$$

$$\vec{H}_{fil} = \sum_{i=1}^{N} \left(\vec{H}_{fil}\right)_i \tag{3}$$

Here $(\vec{H}_{fil})_i$ is the magnetic field created by an imagined filamentary conductor carrying the current I_i. The field $(\vec{H}_{fil})_i$ is obtained from the Biot–Savart law:

$$\left(\vec{H}_{fil}\right)_i = \frac{1}{4\pi}\int_{L_i} \vec{I}_i(\vec{r}', t) \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\, dl \tag{4}$$

Thus the scalar potential in free space obeys the Laplace equation and the boundary integral equation method yields the following surface integral equation [4]:

$$\frac{\phi}{2} + \sum_{i=1}^{N}\int_{S_i} \phi\frac{\partial G}{\partial \vec{n}}\, ds = \sum_{i=1}^{N}\int_{S_i} G\vec{n}\cdot\left(\vec{H}_{fil} - \vec{H}\right) ds \tag{5}$$

Here G is the fundamental solution of the Laplace equation in free space, S_i is the ith conductor's surface, assumed to be smooth, and the unit normal vector $\vec{n}$ is chosen inwards.

Equation (5) includes two unknowns so another relation between the functions ϕ and $\vec{n}\cdot\vec{H}$ is required. This relation should be obtained from the consideration of the problem in the conducting region. Under the condition of the skin effect, it is natural to use the surface impedance boundary condition for this purpose.

III. INTEGRAL EQUATION FORMULATION ENFORCING LEONTOVICH'S IMPEDANCE BOUNDARY CONDITIONS

Let the time variation of the incident field be such that the penetration depth δ into the body remains small as compared with the characteristic size D of the body surface

$$\delta = \sqrt{\tau/(\sigma\mu)} \ll D \tag{6}$$

where τ is the ratio $2/\omega$ for time-harmonic case or duration τ_p of the incident current pulse for transient case. Then the normal component of the magnetic field on the conductor surface can be expressed in terms of tangential components using the well known Leontovich's SIBC's [5] that can be represented in the frequency- and time-domain as follows:

$$\vec{n}\cdot\vec{H} = L\left[\vec{H}\right] \tag{7}$$

Here the operator L is defined for an arbitrary vector function $\vec{f}(\vec{r}, t)$ as follows

$$L\left[\vec{f}\right] = \begin{cases} \dfrac{1-j}{2}\,\delta\nabla_s\cdot\vec{f}\exp(j\omega t) & \text{in frequency domain} \\ (\pi\sigma\mu)^{-1/2}t^{-1/2} * \nabla_s\cdot\vec{f} & \text{in time domain} \end{cases} \tag{8}$$

The asterisk denotes a time-convolution product and the operator of surface divergence $\nabla_s\cdot$ is defined as follows:

$$\nabla_s\cdot\vec{f} = \nabla\cdot\left[\left(\vec{n}\times\vec{f}\right)\times\vec{n}\right] \tag{9}$$

Substituting (2) in (7), we obtain

$$\vec{n}\cdot\vec{H} = \begin{cases} \dfrac{1-j}{2}\,\delta\left(\nabla_s\cdot\vec{H}^{fil} + \nabla_s^2\dot{\phi}\right)\exp(j\omega t) \\ (\pi\sigma\mu)^{-1/2}t^{-1/2} * \left(\nabla_s\cdot\vec{H}_{fil} + \nabla_s^2\phi\right) \end{cases} \tag{10}$$

Relation (10) is substituted into (5) so that the integral equation becomes solvable with respect to ϕ.

Numerical solution of the time domain integral equation involving time convolution product is impractical due to high volume of computer resources required for computation. Therefore implementation of the formulation (5)–(10) is usually performed in the frequency domain. However, in the latter case the formulation unavoidably includes the frequency of the magnetic field source so the integral equation should be re-solved if ω is changed.

On the other hand, taking into account properties of the SIBC (10) leads to the idea of transformation of (5)–(10) to the form admitting separation of variables into spatial and time components in the general case.

IV. Properties of the SIBC

1) The SIBC can be Represented as a Superposition of the Spatial and Time Commutative Operators: We introduce the following spatial and time operators

$$\Psi\left[\vec{f}\right] = \nabla_s\cdot\vec{f} \tag{11a}$$

$$\Omega\left[\vec{f}\right] = \begin{cases} \dfrac{1-j}{2}\,\delta\vec{f}\exp(j\omega t) & \text{in frequency domain} \\ (\pi\sigma\mu)^{-1/2}t^{-1/2} * \vec{f} & \text{in time domain} \end{cases} \tag{11b}$$

Then it is easy to see that

$$\vec{n}\cdot\vec{H} = L\left[\vec{H}\right] = \Psi\left[\Omega\left[\vec{H}\right]\right] = \Omega\left[\Psi\left[\vec{H}\right]\right] \tag{12}$$

From (11a)–(11b) it follows that if $\vec{f}(\vec{r}, t)$ can be represented in the form

$$\vec{f}(\vec{r}, t) = \vec{u}(\vec{r})v(t)$$

then

$$L\left[\vec{f}\right] = \Psi[\vec{u}]\Omega[v] \tag{13}$$

2) The Right Hand Side of the SIBC Includes the Small Parameter: It is a well-known fact that the normal magnetic field and tangential electric field in the skin layer of a conductor is much smaller than the tangential magnetic field. Since in the limiting case $\delta = 0$ the SIBC becomes the condition on the surface of a perfect electrical conductor (PEC), it is natural to expect that the small parameter is proportional to the penetration depth. This property was first discussed by Rytov who used δ as the small parameter in his classical paper on calculation of the skin effect in frequency domain using the perturbation technique [4]. The situation in the time domain is not as evident, therefore rigorous analysis in terms of nondimensional variables is required.

V. BIE-SIBC Formulation in the Invariant Form

A. Non-Dimensional Variables

We introduce the local orthogonal Cartesian coordinate system (ξ_1, ξ_2, η) defined as

$$\vec{e}_{\xi_1}\times\vec{e}_{\xi_2} = \vec{e}_\eta = \vec{n} \tag{14}$$

where $\vec{e}_{\xi_1}, \vec{e}_{\xi_2}, \vec{e}_\eta$ are the unit basis vectors. The characteristic lengths associated with the variables ξ_1, ξ_2 and η are D^* and δ, respectively.

Following the theory of the perturbation methods, we now switch to the dimensionless variables by choosing appropriate scale factors. We introduce the basic scale factors I^*, D^* and τ^* for the current, surface coordinates ξ_1, ξ_2 and time, respectively. The scale factors for other values can be expressed in terms of the basic scale factors. The procedure of derivation of the scale factors has been described in details in [5] so here we give only the results:

$$\left.\begin{array}{ll} H^* = I^*/(4\pi D^*); & \phi^* = I^*/(4\pi) \\ \eta^* = \delta = (\delta/D^*)D^* = pD^*; & p = \left(\tau^* \Big/ \left(\sigma\mu D^{*^2}\right)\right)^{1/2} \ll 1 \end{array}\right\} \tag{15}$$

Clearly, the values $\vec{H}$ and $\vec{H}_i^{fil}$ have the same scale factor.

With the nondimensional variables, (5) and (10) take the form:

$$\frac{\tilde{\phi}}{2} + \sum_{i=1}^{N}\int_{S_i}\tilde{\phi}\frac{\partial G}{\partial\vec{n}}\,d\tilde{s} = \sum_{i=1}^{N}\int_{S_i} G\left(\vec{n}\cdot\tilde{\vec{H}}_{fil} - p\tilde{L}\left[\tilde{\vec{H}}_{fil} - \nabla_s^2\tilde{\phi}\right]\right)d\tilde{s} \tag{16}$$

$$\tilde{L}\left[\vec{f}\right] = \begin{cases} \dfrac{1-j}{2}\,\tilde{\nabla}_s\cdot\dot{\tilde{\vec{f}}}\exp(2j\tilde{t}) \\ \dfrac{1}{\sqrt{\pi\tilde{t}}} * \tilde{\nabla}_s\cdot\tilde{\vec{f}} \end{cases}; \qquad \tilde{\nabla}_s\cdot\tilde{\vec{f}} = \sum_{k=1}^{2}\frac{\partial\tilde{f}_{\xi_k}}{\tilde{\xi}_k}$$

The sign "˜" denotes nondimensional variables. As a result of introduction of the scale factors and transfer to the nondimensional variables, the small parameter p appears in the SIBC and, consequently, in the integral equation (16).

B. Expansions in the Small Parameter

We represent the magnetic scalar potential in the form of the power series in the small parameter p:

$$\tilde{\phi} = \sum_{k=0}^{\infty} p^k\tilde{\phi}_k \tag{17}$$

Substituting the expansions (17) into the formulation (16) and equating the coefficients of equal powers of p, the following integral equations for the first and second coefficients of expansions are obtained:

$$k=0:\quad \frac{\tilde{\phi}_0}{2}+\sum_{i=1}^{N}\int_{S_i}\tilde{\phi}_0\frac{\partial G}{\partial\vec{n}}\,d\tilde{s} = \sum_{i=1}^{N}\int_{S_i}G\vec{n}\cdot\tilde{\vec{H}}_{fil}\,d\tilde{s} \tag{18}$$

$$k=1:\quad \frac{\tilde{\phi}_1}{2}+\sum_{i=1}^{N}\int_{S_i}\tilde{\phi}_1\frac{\partial G}{\partial\vec{n}}\,d\tilde{s} = -\sum_{i=1}^{N}\int_{S_i}G\tilde{L}\left[\tilde{\vec{H}}_{fil}-\nabla\tilde{\phi}_0\right]d\tilde{s} \tag{19}$$

The first equation gives the solution of the problem in the PEC-limit. The second equation gives the correction taking into account the electromagnetic field diffusion into the conductor in the direction normal to the surface of the conductor. As we demonstrate in the next section, both integral equations admit separation of variables.

C. Separation of Variables

Represent $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\tilde{\vec{H}}^{fil}$ in the form:

$$\tilde{\phi}_0(\vec{r},t)=\tilde{\varphi}_0(\vec{r})\tilde{T}_0(t);\quad \tilde{\phi}_1(\vec{r},t)=\tilde{\varphi}_1(\vec{r})\tilde{T}_1(t) \tag{20}$$

$$\tilde{\vec{H}}^{fil}(\vec{r},t)=\tilde{T}_0(t)\frac{1}{4\pi}\sum_{i=1}^{N}\int_{L_i}\tilde{\vec{I}}_i(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,dl = \tilde{T}_0(t)\tilde{\vec{H}}^{r}_{fil}(\vec{r}) \tag{21}$$

where $\tilde{T}_0$ is given by (1) and $\tilde{T}_1=\tilde{\Omega}[\tilde{T}_0]$. Substituting (20)–(21) into (18)–(19) and taking into account (13), we obtain the following integral equations for the spatial functions $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$:

$$\frac{\tilde{\varphi}_0}{2}+\sum_{i=1}^{N}\int_{S_i}\tilde{\varphi}_0\frac{\partial G}{\partial\vec{n}}\,d\tilde{s}=\sum_{i=1}^{N}\int_{S_i}G\vec{n}\cdot\tilde{\vec{H}}^{r}_{fil}\,d\tilde{s} \tag{22}$$

$$\frac{\tilde{\varphi}_1}{2}+\sum_{i=1}^{N}\int_{S_i}\tilde{\varphi}_1\frac{\partial G}{\partial\vec{n}}\,d\tilde{s}=-\sum_{i=1}^{N}\int_{S_i}G\Psi\left[\tilde{\vec{H}}^{r}_{fil}-\nabla\tilde{\varphi}_0\right]d\tilde{s} \tag{23}$$

Let us emphasize the main advantages of the formulation developed:

1) The form of the integral equations, including the right hand side, is independent of the time dependence of the magnetic field source and is determined solely by the geometric parameters of the given system of conductors. Therefore, by solving the integral equations just once for a given system of conductors and multiplying the result by the corresponding time function, one can obtain solutions for any time dependence of the current.
2) Integral equations (22) and (23) differ only in the form of the right hand side and can be solved by the same programmed routine; therefore, new computational complications do not arise beyond those involved in solving the problem in well-known PEC-limit.

VI. SIBC's of High Order of Approximation

The Leontovich SIBC (7)–(8) is the condition of low order approximation since it does not take into account the following important factors: the curvature of the body surface and the field diffusion in the direction tangential to the body surface. SIBC of *high* order of approximation, allowing for these factors, can be written in the form [6]:

$$\begin{aligned}\vec{n}\cdot\dot{\vec{H}} &= -\dot{H}_\eta\\ &= -\sum_{k=1}^{2}\Bigg[\frac{1-j}{2}\,\delta\frac{\partial\dot{H}_{\xi_k}}{\partial\xi_k}-\frac{j}{4}\,\frac{d_{3-k}-d_k}{d_kd_{3-k}}\,\delta^2\frac{\partial\dot{H}_{\xi_k}}{\partial\xi_k}\\ &\quad-\frac{1+j}{32}\,\frac{3d_{3-k}^2-d_k^2-2d_kd_{3-k}}{d_k^2d_{3-k}^2}\\ &\quad\times\delta^3\frac{\partial\dot{H}_{\xi_k}}{\partial\xi_k}-\frac{1+j}{8}\\ &\quad\times\delta^3\left(-\frac{\partial^3\dot{H}_{\xi_k}}{\partial\xi_k\partial\xi_{3-k}^2}+\frac{\partial^3\dot{H}_{\xi_k}}{\partial\xi_k^3}+2\frac{\partial^3\dot{H}_{\xi_{3-k}}}{\partial\xi_k^2\partial\xi_{3-k}}\right)\Bigg]\end{aligned} \tag{24a}$$

$$\begin{aligned}\vec{n}\cdot\vec{H} &= -H_\eta\\ &= -\sum_{k=1}^{2}\Bigg[\frac{1}{\sqrt{\pi\sigma\mu t}}*\frac{\partial H_{\xi_k}}{\partial\xi_k}+\frac{d_{3-k}-d_k}{2d_kd_{3-k}}\,\frac{U(t)}{\sigma\mu}*\frac{\partial H_{\xi_k}}{\partial\xi_k}\\ &\quad+\frac{3d_{3-k}^2-d_k^2-2d_kd_{3-k}}{8d_k^2d_{3-k}^2}\sqrt{\frac{t}{\pi\sigma^3\mu^3}}\\ &\quad*\frac{\partial H_{\xi_k}}{\partial\xi_k}+\frac{1}{2}\sqrt{\frac{t}{\pi\sigma^3\mu^3}}\\ &\quad*\left(-\frac{\partial^3H_{\xi_k}}{\partial\xi_k\partial\xi_{3-k}^2}+\frac{\partial^3H_{\xi_k}}{\partial\xi_k^3}+2\frac{\partial^3H_{\xi_{3-k}}}{\partial\xi_k^2\partial\xi_{3-k}}\right)\Bigg]\end{aligned} \tag{24b}$$

Here $U(t)$ is the unit step function and d_d, $k=1,2$, are the local radii of curvature of the corresponding coordinate line ξ_k. Clearly, the SIBC of low order is included in the SIBC of high order.

It can be demonstrated that the proposed technique is applicable to the SIBC (24). Below we describe the final result, namely: the technique in terms of dimensional variables for calculation of the distribution of the scalar potential over surfaces of the conductors.

1) Find the spatial functions $\varphi_n(\vec{r})$ from the solution of the following integral equations:

$$\frac{\varphi_n}{2}+\sum_{i=1}^{N}\int_{S_i}\varphi_n\frac{\partial G}{\partial\vec{n}}\,ds=\sum_{i=1}^{N}\int_{S_i}GR_n\,ds;\quad n=0,1\cdots3 \tag{25}$$

$$R_0(\vec{r})=\vec{n}\cdot\vec{H}^{r}_{fil}(\vec{r}) \tag{26a}$$

$$R_1(\vec{r})=-\sum_{k=1}^{2}\partial\left(\vec{H}_0\right)_{\xi_k}\Big/\partial\xi_k \tag{26b}$$

$$R_2(\vec{r})=-\sum_{k=1}^{2}\left[\frac{\partial\left(\vec{H}_1\right)_{\xi_k}}{\partial\xi_k}+\frac{d_{3-k}-d_k}{2d_kd_{3-k}}\,\frac{\partial\left(\vec{H}_0\right)_{\xi_k}}{\partial\xi_k}\right] \tag{26c}$$

$$R_3(\vec{r}) = -\sum_{k=1}^{2}\left[\frac{\partial\left(\vec{H}_2\right)_{\xi_k}}{\partial\xi_k} + \frac{d_{3-k}-d_k}{2d_kd_{3-k}}\frac{\partial\left(\vec{H}_1\right)_{\xi_k}}{\partial\xi_k} + \frac{3d_{3-k}^2-d_k^2-2d_kd_{3-k}}{8d_k^2d_{3-k}^2}\frac{\partial\left(\vec{H}_0\right)_{\xi_k}}{\partial\xi_k} + \frac{1}{2}\left(-\frac{\partial^3\left(\vec{H}_0\right)_{\xi_k}}{\partial\xi_k\partial\xi_{3-k}^2} + \frac{\partial^3\left(\vec{H}_0\right)_{\xi_k}}{\partial\xi_k^3} + 2\frac{\partial^3\left(\vec{H}_0\right)_{\xi_{3-k}}}{\partial\xi_k^2\partial\xi_{3-k}}\right)\right] \tag{26d}$$

Here
$\left(\vec{H}_0\right)_{\xi_k} = \left(\vec{H}^r_{fil}\right)_{\xi_k} - \partial\varphi_0/\partial\xi_k$; $\left(\vec{H}_1\right)_{\xi_k} = -\partial\varphi_1/\partial\xi_k$; $\left(\vec{\tilde{H}}_2\right)_{\xi_k} = -\partial\varphi_2/\partial\xi_k$

2) Calculate time functions $T_n(t)$ corresponding to a given time dependence of the magnetic field source:

$$T_1 = \begin{cases} ((1-j)/2)\delta T_0\exp(j\omega t) \\ T_0 * (\pi\sigma\mu t)^{-1/2} \end{cases} \tag{27a}$$

$$T_2 = \begin{cases} -(j/2)\delta^2 T_0\exp(j\omega t) \\ (\sigma\mu)^{-1/2}T_0 * U \end{cases} \tag{27b}$$

$$T_3 = \begin{cases} -((1+j)/4)\delta^3 T_0\exp(j\omega t) \\ T_0 * (t/(\pi\sigma^3\mu^3)^{1/2} \end{cases} \tag{27c}$$

3) Find the scalar potential using the following formula:

$$\phi(\vec{r}, t) = \sum_{n=0}^{3}\varphi_n(\vec{r})T_n(t) \tag{28}$$

If we restrict ourselves by the first term in (28), we obtain the solution of the problem in the PEC-limit. The second, third and fourth terms give the corrections of the order of the Leontovich, Mitzner and Rytov approximations, respectively. The method of selection the order of the approximation that is best suited for a given problem is given in [7].

It is easy to see that formulation (25)–(28) keeps all advantages of the formulation (20)–(23) discussed in the previous section.

VII. Conditions of Applicability of the Techinque

1) The technique can not be applied if (1) is not satisfied. It means that sources of the magnetic field in the problem considered must be correlated in time. Otherwise the variables may be separated only in the equation for the zero-order term (the PEC-limit)
2) Velocity of propagation of the field is considered as infinite. In other words, the technique is not applicable for high-frequency problems where the displacement current should be taken into account.
3) The technique can not be applied to nonlinear problems.

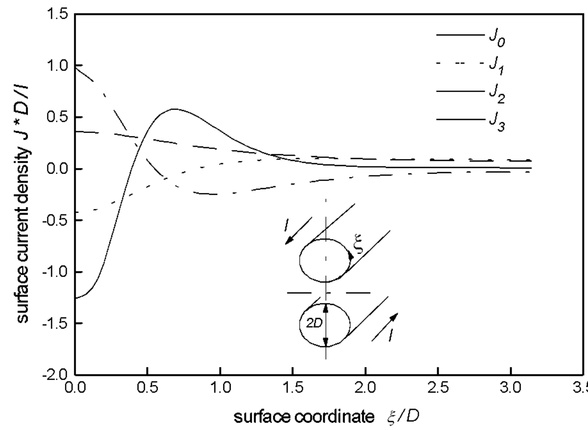

Fig. 1. Spatial coefficients of expansions of the surface current density.

VIII. Numerical Example

We calculated the distribution of the surface current density $\vec{J} = \vec{n} \times \vec{H}$ for a pair of identical parallel copper conductors with circular cross section where equal and opposite directed single trapezoidal pulses are flowing from an external source. The radius of each conductor was taken equal to the distance between them. Vector $\vec{J}$ is directed along the conductor and can be treated as scalar. Fig. 1 shows the distributions of the spatial coefficients $J_n^r(\vec{r})$, $n = 0, 1\cdots 3$ along half the cross section contour of one conductor. It is noted than every coefficient acts in the direction to smooth out nonuniformities in the distribution of the previous terms of expansions.

IX. Conclusions

The technique of separation of variables into spatial and time components in the integral equation formulations enforcing the surface impedance boundary conditions has been proposed for any time dependence of the magnetic field source. Thus the integral equations depend only on the geometry of the problem and maintain the same form in the time- and frequency domains.

References

[1] T. H. Fawzi, M. T. Ahmed, and P. Burke, "On the use of the impedance boundary condition in eddy current problems," *IEEE Trans. on Magnetics.*, vol. 21, no. 5, pp. 1835–1840, 1985.
[2] K. Ishibashi, "Eddy current analysis by boundary element method utilizing impedance boundary condition," *IEEE Trans. on Magnetics.*, vol. 31, no. 3, pp. 1500–1503, 1995.
[3] S. Yuferev, "Generalized BIE-BLA formulation of skin and proximity effect problems for an arbitrary regime of the magnetic field generation," in *Proc. 2d IEE CEM Conf.*: IEE-384, pp. 307–310.
[4] S. M. Rytov, "Calculation of skin effect by perturbation method," *Zhurnal Experimental'noi I Teoreticheskoi Fiziki*, vol. 10, pp. 180–189, 1940.
[5] S. Yuferev and L. Kettunen, "A unified surface impedance concept for both linear and nonlinear skin effect problems," *IEEE Trans. on Magnetics*, vol. 35, no. 3, pp. 1454–1457, 1999.
[6] S. Yuferev and N. Ida, "Time domain surface impedance boundary conditions of high order of approximation," *IEEE Trans. on Magnetics.*, vol. 34, pp. 2605–2608, 1998.
[7] ——, "Selection of the surface impedance boundary conditions for a given problem," *IEEE Trans. on Magnetics*, vol. 35, no. 3, pp. 1486–1489, 1999.