Use of the Perturbation Technique for Implementation of Surface Impedance Boundary Conditions for the FDTD Method

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***Abstract*—A new approach to implement a surface impedance boundary condition (SIBC) for the FDTD method is proposed. The explicit FDTD formulation for the boundary cell closest to the surface of conducting body is obtained using the perturbation technique in the small parameter proportional to the ratio of the values of the electric field at the opposite sides of the cell. The condition of applicability of the technique is obtained. It is shown that an FDTD code where the PEC-condition has been already implemented can be easily upgraded using the proposed technique to take into account properties of the conducting medium and improve accuracy of the computation. Numerical examples are included to illustrate the theory.**

***Index Terms*—FDTD, surface impedance boundary condition, skin effect, perturbation methods, time domain analysis.**

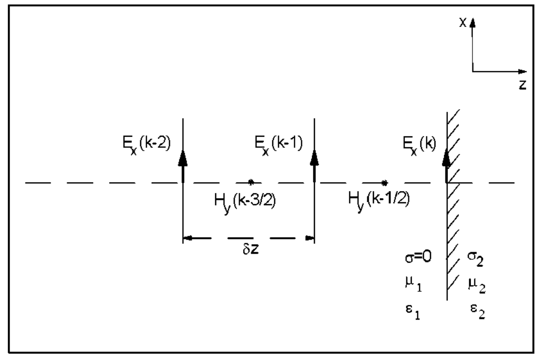

Fig. 1. One-dimensional FDTD grid.

I. Introduction

THE goal of the surface impedance concept is to provide approximate relations between tangential electric and magnetic fields on the surface of the conductor (lossy dielectric) under the condition of skin effect. Then the conducting region may be replaced by the surface impedance boundary conditions (SIBC's) and eliminated from the numerical procedure [1], [2]. The original frequency domain conditions for homogeneous body (well-known Leontovich's SIBC) have been transformed to the time domain form and extended to cover nonhomogeneous and nonlinear media [3], [4]. Recently, time domain SIBC's of high order of approximation have been developed [5]. At present, different SIBC's are frequently used in combination with the FDTD method to restrict the computational grid to the surface of the conducting region [6]–[9].

All time domain SIBC's contain time-convolution integrals that makes their implementation rather complicated because the finite difference algorithm becomes implicit for the boundary cell whereas it remains explicit for other cells of the grid. To compute efficiently the time convolution integral in Leontovich's SIBC, Oh and Schutt-Aine [7] approximated the impedance function of the lossy dielectric medium with a series of first-order rational functions. Finally it provided explicit formulation for the boundary cell. Although this approach can be applied to the SIBC's with other impedance functions, a new approximation should be done in every case. In this paper we propose another way based on the fact that the electric field on the surface of a lossy dielectric body is less than in the pure dielectric medium near the surface. Thus the FDTD formulation for the boundary cell can be transformed using the perturbation technique in the small parameter proportional to the ratio of the values of the electric field at the opposite sides of the cell. Note that no consideration of the impedance function is now required to obtain explicit formulation for the boundary cell. So this method seems more general since it is suitable for all SIBC's without any preliminary approximations.

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II. Finite Difference Equations

Without any loss of generality and to symplify mathematical description, we consider a one-dimensional problem of an electromagnetic pulse travelling in nonconducting space normal to the plane infinite surface of the lossy dielectric medium shown in Fig. 1. Then the Maxwell equations for nonconducting region take the following form:

$$\partial H_y/\partial t = -\mu_1^{-1}\partial E_x/\partial z \tag{1}$$

$$\partial E_x/\partial t = -\varepsilon_1^{-1}\partial H_y/\partial z \tag{2}$$

Since the electric and magnetic fields have only one component, subscripts "x" and "y" can be omitted.

The electric and magnetic fields on the interface between two media are related by a SIBC that can be written in the following general form:

$$E = f(H) \tag{3}$$

where the time domain impedance function f is known.

Finite difference equations for the boundary k-cell at the n-time step can be obtained from (1), (2) and written in the form of Yee's algorithm:

$$H_{k-1/2}^{n+1/2} = H_{k-1/2}^{n-1/2} + \delta t(\mu_1 \delta z)^{-1}(E_{k-1}^n - E_k^n) \tag{4}$$
$$E_{k-1}^{n+1} = E_{k-1}^n - \delta t(\varepsilon_1 \delta z)^{-1}(H_{k-1/2}^{n+1/2} - H_{k-3/2}^{n+1/2}) \tag{5}$$

The SIBC (3) in the discrete space and time can be written in the form:

$$E_k^n \approx f(H_{k-1/2}^{n+1/2}) \tag{6}$$

Substituting (6) into (4), we obtain:

$$H_{k-1/2}^{n+1/2} + \frac{\delta t}{\mu_1 \delta z} f(H_{k-1/2}^{n+1/2}) = H_{k-1/2}^{n-1/2} + \frac{\delta t}{\mu_1 \delta z} E_{k-1}^n \tag{7}$$

Since the function f involves the time convolution product, (7) cannot be solved with respect to $H_{k-1/2}^{n+1/2}$ by analytical methods. This makes the algorithm implicit for the "boundary" cell.

III. Perturbation Technique

Since E_k^n is related to the interface whereas E_{k-1}^n is related to the dielectric medium, we can assume that

$$E_k^n \ll E_{k-1}^n \tag{8}$$

Thus (4) contains terms of different orders of magnitude so we can transform this equation using the perturbation technique. As the first step we transfer to nondimensional variables by choosing appropriate scale factors.

Let E_{k-1}^* be the characteristic scale for variation of E_{k-1}^n so that we can write

$$E_{k-1}^n = \tilde{E}_{k-1}^n E_{k-1}^* \tag{9}$$

Here and below the sign "∼" denotes nondimensional values. Using (8), the scale factor for E_k^n can be represented in the form:

$$E_k^* = dE_{k-1}^*; \tag{10}$$
$$d = E_k^*/E_{k-1}^* \ll 1 \tag{11}$$

where d is the small parameter. Thus we obtain

$$E_k^n = \tilde{E}_k^n E_k^* = d\tilde{E}_k^n E_{k-1}^* \tag{12}$$

Since quantities $H_{k-1/2}^{n+1/2}$ and $H_{k-1/2}^{n-1/2}$ are of the same order of magnitude, they should have a common scale factor $H_{k-1/2}^*$ that can be defined from (4) as follows:

$$H_{k-1/2}^* = E_{k-1}^* \delta t/(\mu_1 \delta z) \tag{13}$$

Using (13), nondimensional values $\tilde{H}_{k-1/2}^{n+1/2}$ and $\tilde{H}_{k-1/2}^{n-1/2}$ can be represented in the form:

$$H_{k-1/2}^{n+1/2} = H_{k-1/2}^*$$
$$\tilde{H}_{k-1/2}^{n+1/2} = \tilde{H}_{k-1/2}^{n+1/2} E_{k-1}^* \delta t/(\mu_1 \delta z) \tag{14a}$$
$$H_{k-1/2}^{n-1/2} = H_{k-1/2}^*$$
$$\tilde{H}_{k-1/2}^{n-1/2} = \tilde{H}_{k-1/2}^{n-1/2} E_{k-1}^* \delta t/(\mu_1 \delta z) \tag{14b}$$

With the nondimensional variables, (4) takes the form:

$$\tilde{H}_{k-1/2}^{n+1/2} = \tilde{H}_{k-1/2}^{n-1/2} + (\tilde{E}_{k-1}^n - d\tilde{E}_k^n) \tag{15}$$

We represent the function $\tilde{H}_{k-1/2}^{n+1/2}$ for which the solution is sought in the form of expansions in the small parameter d:

$$\tilde{H}_{k-1/2}^{n+1/2} = \tilde{h}_0 + d\tilde{h}_1 + O(d^2) \tag{16}$$

Substituting (16) into (15) and equating the coefficients of equal powers of d, we obtain equations for $\tilde{h}_0$ and $\tilde{h}_1$:

$$\tilde{h}_0 = \tilde{H}_{k-1/2}^{n-1/2} + \tilde{E}_{k-1}^n; \tag{17a}$$
$$\tilde{h}_1 = -\tilde{E}_k^n = -\tilde{f}(\tilde{h}_0) \tag{17b}$$

Therefore, we developed a two-step technique: first $\tilde{h}_0$ is calculated in (17a) and then $\tilde{h}_1$ is calculated in (17b).

With the dimensional variables, formulation (17) takes the form:

$$h_0 = H_{k-1/2}^{n-1/2} + \frac{\delta t}{\mu_1 \delta z} E_{k-1}^n \quad (14a) \tag{18a}$$
$$h_1 = -\frac{\delta t}{\mu_1 \delta z} f(h_0) \tag{18b}$$
$$H_{k-1/2}^{n+1/2} = h_0 + h_1 \tag{19}$$

It is easy to see that the formulation (18), (19) is explicit because the term containing the impedance function is now on the right hand side of (18b).

Note that we do not actually know the exact values of d and E_{k-1}^*. Both of them are required only at the stage of derivation and they are obviously not included in the final formulation (18), (19). All we really need to know is that condition (8) is satisfied (and, consequently, (11) holds).

Let us emphasize that the representation in (19) has clear physical meaning, namely: h_0 is the magnetic field calculated under the assumption that medium 2 is a perfect electrical conductor (PEC) and the electric field at the interface is zero; h_1 is the correction allowing for finite conductivity of medium 2. Therefore, an FDTD code where the PEC-condition has been implemented can be easily upgraded by adding (18b) to the formulation for the boundary cell.

IV. Condition of Applicability of the Technique

We will perform the derivation in the frequency domain considering propagation of a uniform plane wave from region 1 ($z < 0$) to region 2 ($z > 0$) (Fig. 1). Because of reflection, the following waves travel in both regions:

$$\text{Incident wave:} \quad E_1^+ = E_{10}^* \exp(-j\beta_1 z) \tag{20a}$$
$$\text{Reflected wave:} \quad E_1^- = E_{10}^- \exp(j\beta_1 z) \tag{20b}$$
$$\text{Transmitted wave:} \quad E_2^+ = E_{20}^+ \exp(-\gamma_2 z) \tag{20c}$$

where

$$\beta_1 = \omega\sqrt{\mu_1 \varepsilon_1} = 2\pi/\lambda; \quad \gamma_2 = \sqrt{(\sigma_2 + j\omega)j\omega\mu_2}$$

Since the total electric field is continuous at $z = 0$, the reflection coefficient can be represented in the form:

$$\Gamma = E_{10}^{-}/E_{10}^{+} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_2/\eta_1 - 1}{\eta_2/\eta_1 + 1}$$
$$\eta_1 = \sqrt{\mu_1/\varepsilon_1}; \quad \eta_2 = \sqrt{j\omega\mu_2/(\sigma_2 + j\omega\varepsilon_2)} \tag{21}$$

It is natural to assume that

$$\sigma_2 \gg \omega\varepsilon_2 \quad \text{so that} \quad \eta_2/\eta_1 \ll 1 \tag{22}$$

Then the reflection coefficient can be represented in the form:

$$\Gamma = (s-1)(1-s) + O(s^2) = -1 + 2s + O(s^2) \tag{21}$$

where

$$s = \sqrt{\frac{j\omega\mu_2\varepsilon_1}{\sigma_2\mu_1}} = \sqrt{\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}\sqrt{\frac{j\omega\varepsilon_2}{\sigma_2}} = \frac{1+j}{\sqrt{2}}\, ck_2^{-1} \ll 1$$
$$k_2 = \sqrt{\frac{\sigma_2}{\omega\varepsilon_2}} = \sqrt{\tan\theta}; \quad c = \sqrt{\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}$$

where $\tan\theta$ is the loss tangent of medium 2.

Using (21), we write the total electric field in medium 1 in the form:

$$\begin{aligned} E_1(z) &= E_{10}^{+}\exp(-j\beta_1 z) + E_{10}^{-}\exp(j\beta_1 z) \\ &= E_{10}^{+}(\exp(-j\beta_1 z) + \Gamma\exp(j\beta_1 z)) \\ &= E_{10}^{+}[\exp(-j\beta_1 z) + (-1 + \sqrt{2}ck_2^{-1} + j\sqrt{2}_2^{-1}) \\ &\quad \cdot \exp(j\beta_1 z)] \\ &= E_{10}^{+}\sqrt{2}ck_2^{-1}\{[\cos\beta_1 z - \sin\beta_1 z] \\ &\quad + j[(-\sqrt{2}ck_2 + 1)\sin\beta_1 z + \cos\beta_1 z]\} \end{aligned} \tag{22}$$

Setting $z = 0$ in (22), we obtain the total field on the interface:

$$E_1(0) = E_{10}^{+}\sqrt{2}ck_2^{-1}(1+j) \tag{23}$$

The ratio of (21) and (22) gives the variation of the field in medium 1 with respect to the field on the interface:

$$\begin{aligned} \frac{E_1(z)}{E_1(0)} = &\left\{\left[2\cos\frac{2\pi}{\lambda_1}z - \sqrt{2}ck_2\sin\frac{2\pi}{\lambda_1}z\right]\right. \\ &\left. + j(-\sqrt{2}ck_2 + 2)\sin\frac{2\pi}{\lambda_1}z\right\} \end{aligned} \tag{24}$$

Our goal is evaluation of the electric fields on the opposite sides of the "boundary" cell (see Fig. 1). Usually, more than 10 cells per wavelength are taken in computations. Thus

$$2\pi z/\lambda_1 \ll 1 \tag{25}$$

Expanding $\sin(2\pi/\lambda_1)z$ and $\cos(2\pi/\lambda_1)z$ in a Taylor series and taking into account (25), we represent (24) in the form:

$$\frac{E_1(z)}{E_1(0)} \approx \left\{\left[2 - \sqrt{2}ck_2\frac{2\pi}{\lambda_1}z\right] + j(-\sqrt{2}ck_2 + 2)\frac{2\pi}{\lambda_1}z\right\} \tag{26}$$

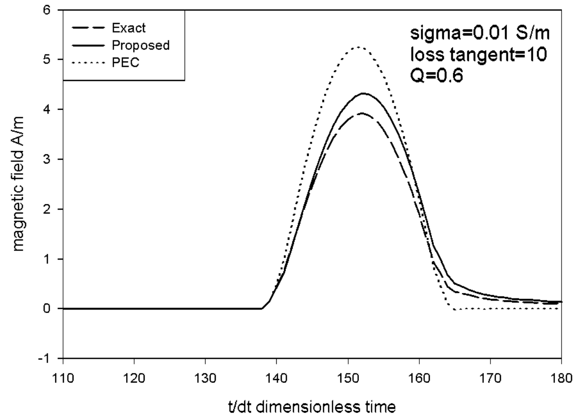

Fig. 2. Distribution of the magnetic field at the interface (1-D case).

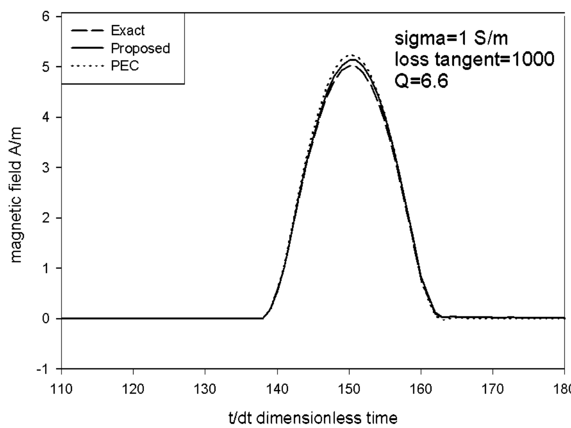

Fig. 3. Distribution of the magnetic field at the interface (1-D case).

Setting $z = \delta z$, we obtain:

$$\frac{E_1(\delta z)}{E_1(0)} \gg 1 \quad \text{until}$$
$$Q = \sqrt{\tan\theta}\sqrt{\frac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}}\frac{2\pi}{\lambda_1}\delta z \gg 1 \tag{27}$$

The error ε due to the representation in (18) is derived directly from the condition in (27) and written in the form:

$$\begin{aligned} \varepsilon &= Q^{-2} = (\tan\theta)^{-1}\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\left(\frac{\lambda_1}{2\pi\delta z}\right)^2 \\ &= \frac{\mu_2}{\omega\sigma_2(\mu_1\delta z)^2} \end{aligned} \tag{28}$$

V. Numerical Examples

The formulation (18), (19) has been used for implementation of the following time domain Leontovich SIBC for 1-D and 2-D FDTD codes:

$$\begin{aligned} \vec{n} \times \vec{E} &= T * ((\vec{n} \times \vec{H}) \times \vec{n}) \\ T(t) &= \left(\frac{\mu}{\varepsilon}\right)^{1/2}\left\{\delta(t) + \frac{\sigma}{2\varepsilon}\left[I_1\left(\frac{\sigma t}{2\varepsilon}\right) - I_0\left(\frac{\sigma t}{2\varepsilon}\right)\right]\right. \\ &\quad \left. \cdot \exp\left(-\frac{\sigma t}{2\varepsilon}\right)\right\} \end{aligned} \tag{29}$$

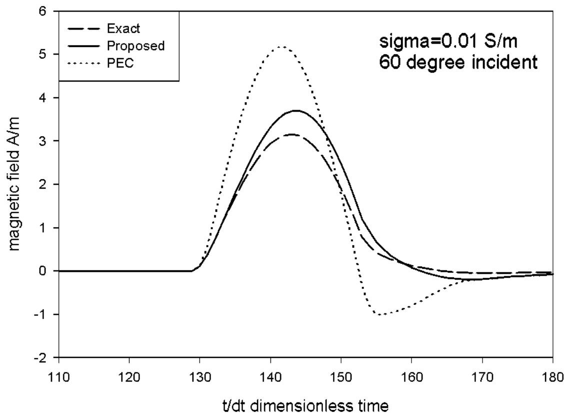

Fig. 4. Distribution of the magnetic field at the interface (2-D case).

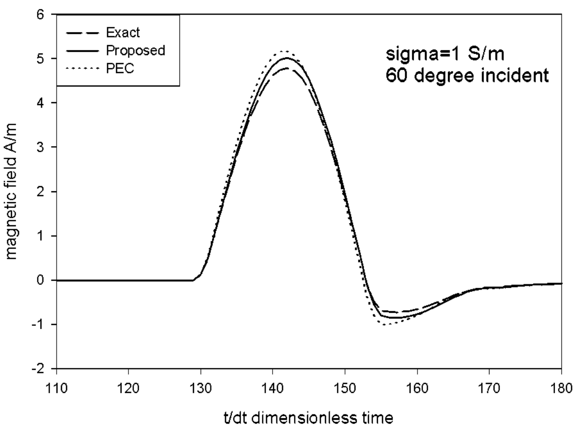

Fig. 5. Distribution of the magnetic field at the interface (2-D case).

where "$*$" denotes the time-convolution product, $I_n(x)$ is the modified Bessel function of order n and $\delta(t)$ is the Dirac function.

A. One-Dimensional Case

A normally incident TM plane wave with the following material parameters of a dielectric half-space has been considered:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_0; \quad \mu_1 = \mu_2 = \mu_0; \quad \sigma_1 = 0; \quad \sigma_2 \neq 0;$$

$$\text{Excitation} = 1000 \ \sin(2\pi t/5.56E-8);$$

$$(0 \ < \ t \ < \ (5.56E-8)/2 \text{ sec}$$

The following parameters of the mesh were used:

$$\delta z = 5 \cdot 10^{-1} \text{ m}; \quad \delta t = 1.18 \cdot 10^{-19} \text{ sec}$$

$$N_{z\text{-space steps}} = 100; \quad N_{\text{time steps}} = 300$$

Fig. 2 demonstrates distributions of the magnetic field at the interface obtained for the material with low loss tangent ($\tan\theta = 10$). The curves were obtained using the PEC-condition (dotted line), exact formula (7) (dashed line) and the proposed formulation (18), (19) (solid line). The error is significant as it is given by (28) ($Q \ < \ 1$). For the material with high loss tangent ($\tan\theta = 10^3$) the 2.4% error (Fig. 3) is in agreement with the formulation ($\varepsilon = Q^{-2} = 2.2\%$).

B. Two-Dimensional Case

We considered the problem of oblique incidence of a TM pulse at plane boundary. We used the same parameters for the material and excitation as in the 1-D case. The following parameters of the mesh have been used:

$$\delta x = \delta z = 5 \cdot 10^{-1} \text{ m}; \quad N_{z\text{-space steps}} = N_{x\text{-space steps}} = 100$$

The incident angle was equal to 60 degree. The magnetic field was computed in the middle of the dielectric surface for both the exact and proposed methods. Figs. 4 and 5 demonstrate the results obtained for materials with low- and high loss tangent, respectively.

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