# Surface Impedance Boundary Conditions Near Corners and Edges: Rigorous Consideration

Sergey Yuferev, Leonid Proekt, and Nathan Ida, *Senior Member, IEEE*

***Abstract*—The two-dimensional problem of the magnetic field distribution inside and outside an imperfectly conducting, 90-degree edge is solved using the perturbation technique. The well-known solution of the problem for a perfectly conducting edge is used as an initial approximation. The surface impedance near the edge is represented in the form of asymptotic expansions in the small parameter proportional to the ratio of the skin depth and characteristic size of the conductor surface. An analytical solution for the electric field in the conductor near the edge is obtained. The solution behavior is investigated and the approximate boundary conditions relating the tangential components of the electric and magnetic fields on the conductor surface near the edge are proposed in a form suitable for numerical implementation.**

***Index Terms*—Approximate boundary conditions, asymptotic expansions, edges, perturbation techniques, surface impedance.**

## I. INTRODUCTION

THE SURFACE impedance concept has been originally developed under the assumption that the electromagnetic field distribution in the conductor may be described by a 1-D equation in the direction normal to the conductor's surface. Clearly, this approximation is valid only for smooth objects. To modify the surface impedance boundary condition (SIBC) near a 90° conducting wedge, infinitely long in $z$-direction (2-D problem), Deeley [1] postulated that the transverse magnetic field can be described in terms of the superposition of two modes:

$$H_x = H_{xo}\exp(-\zeta y); \quad H_y = H_{yo}\exp(-\zeta x); \quad \zeta^2 = -j\omega\sigma\mu \tag{1}$$

where $H_{xo}$ and $H_{yo}$ are assumed to be constant. Jingguo and Lavers [2] generalized Deeley's approach for an arbitrary angle corner. However, rigorous consideration of this problem requires the field equations being solved in both the conducting and nonconducting regions near the corner. It is well-known that in the classical limiting case of a perfect conductor, the distribution of the magnetic field along the surface of the edge is singular and can be described by the following function [3]–[6]:

$$H_\rho = A\rho^\gamma \tag{2}$$

Manuscript received June 4, 2000.

S. Yuferev was with The University of Akron, Akron, OH 44325 USA, on leave from Ioffe Physical-Technical Institute, Russian Academy of Sciences (e-mail: sergey.yuferev@nokia.com).

L. Proekt and N. Ida are with The University of Akron, Akron, OH 44325 USA (e-mail: {lp3; ida}@uakron.edu).

Publisher Item Identifier S 0018-9464(01)07846-3.

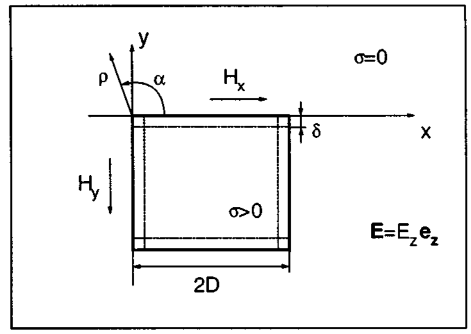


Fig. 1. Geometry of the problem.

where
- $A$ is a known coefficient,
- $\rho$ and $\alpha$ are polar coordinates (see Fig. 1), and
- $\gamma$ $= -1/3$ for a 90° edge.

It is natural to suppose that discontinuities also occur near an imperfectly conducting wedge. Then the assumption in (1) is not correct because the singularity is ignored.

In the present paper we apply the approach previously used to obtain the SIBC's of high order of approximation, namely: perturbation technique in the small parameter equal to the ratio of the skin depth $\delta$ and characteristic size $D$ of the conducting domain. The solution of the problem for the perfect conductor is the zero-order approximation. In the first approximation, the distribution of the magnetic field along the perfectly conducting edge is used as a boundary condition for the field diffusion problem inside the conductor. Thus the solution of this problem gives the correction to the zero-order approximation by taking into account the finite conductivity of the body. In other words, we seek generalization of the solution for the perfectly conducting edge.

## II. STATEMENT OF THE PROBLEM

We consider an eddy current problem of a rectangular conductor in the external electric field directed along the $z$-axis so that the magnetic field has only $x$- and $y$-components (Fig. 1). Assume that the dimension $D$ of the conductor remains small compared to the wavelength $\lambda$. In this case the distributions of the electric and magnetic fields in the conducting and nonconducting regions can be described by the Maxwell equations in the following form:

$$Conductor: \quad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \sigma E; \quad j\omega\mu H_x = \frac{\partial E}{\partial y}; \quad j\omega\mu H_y = -\frac{\partial E}{\partial x} \tag{3a}$$

$$Dielectric:$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0; \quad j\omega\mu H_x = \frac{\partial E}{\partial y}; \quad j\omega\mu H_y = -\frac{\partial E}{\partial x} \tag{3b}$$

$$Boundary\ conditions:$$
$$\vec{H}^{cond}\Big|_{surf} = \vec{H}^{diel}\Big|_{surf}; \quad \vec{E}^{cond}\Big|_{surf} = \vec{E}^{diel}\Big|_{surf}. \tag{3c}$$

Let the parameters of the incident field be such that the electromagnetic penetration depth $\delta$ remains small compared to the characteristic dimension $D$ of the conductor:

$$\delta = \sqrt{2/(\omega\sigma\mu)} \ll D. \tag{4}$$

We eek the surface impedance in the region near an edge i.e., we wish to calculate the following functions:

$$Z_x = \left[\frac{E_z}{H_x}\right]_{\alpha=2\pi}; \quad Z_y = \left[\frac{E_z}{H_y}\right]_{\alpha=3\pi/2} \qquad \text{for } \rho \ll D \tag{5}$$

where $\rho$ is the distance from the edge.

## III. Perturbation Technique

Following the theory of perturbation methods, we now rewrite the problem in terms of dimensionless variables by choosing appropriate scale factors. Let $I^*$ and $\delta$ be scale factors for the current and the polar coordinate $\rho$ (and both Cartesian coordinates $x$ and $y$) respectively.

Then scale factors $H^*$ and $E^*$ for the magnetic and electric fields, respectively, are introduced as follows [7], [8]:

$$H^* = I^*/(4\pi D); \quad E^* = \mu\omega I^*/(8\pi). \tag{6}$$

With the nondimensional variables, (3a) take the form:

$$\tilde{E} = p\left(\frac{\partial \tilde{H}_y}{\partial \tilde{x}} - \frac{\partial \tilde{H}_x}{\partial \tilde{y}}\right); \quad pk^2\tilde{H}_x = \frac{\partial \tilde{E}}{\partial \tilde{y}}; \quad pk^2\tilde{H}_y = -\frac{\partial \tilde{E}}{\partial \tilde{x}} \tag{7}$$

$$p = \delta/D = \sqrt{2/(\omega\sigma\mu D^2)} \ll 1$$

where $k^2 = 2j$ and the parameter $p$ is small since it is proportional to the ratio of the skin depth and characteristic size $D$ of the conductor. "$\sim$" denotes nondimensional variables. Substituting the scale factors into (2) yields:

$$\tilde{H}_\rho \frac{I^*}{4\pi D} = A\tilde{\rho}^\gamma (pD)^\gamma \quad \Leftrightarrow \quad \tilde{H}_\rho = p^\gamma \tilde{\rho}^\gamma \frac{4\pi A}{I^* D^{-\gamma-1}}. \tag{8}$$

Introducing the following scale factor for $A$,

$$A^* = I^* D^{-\gamma-1}/(4\pi)$$

we represent (2) in the form:

$$\tilde{H}_\rho = p\,\tilde{\rho}^\gamma \tilde{A}. \tag{9}$$

As a result of transfer to nondimensional variables, the small parameter $p$ appears in the governing equations. The sign "$\sim$" will be omitted in subsequent expressions.

We seek the magnetic and electric fields in the conducting region in the form of a power series in the small parameter $p$:

$$\vec{H} = p^\gamma \vec{H}_0 + p^{\gamma+1}\vec{H}_1 + \cdots; \quad E = p^\gamma E_0 + p^{\gamma+1}E_1 + \cdots. \tag{10}$$

Since $\gamma + 1 > 0$ (recall that $\gamma < 0$) the terms in (10) beginning from the second term vanish when $p = 0$. Thus the first terms $H_0$ and $E_0$ in (10) describe the field distribution in the perfect electrical conductor limit whereas the others take into account the field diffusion into the conductor.

Substituting the expansions (10) into (7) and (9) and equating the coefficients of equal powers of $p$, we obtain the following equations for the coefficients in (10):

$\gamma$:

$$E_0 = 0; \tag{11a}$$

$$\left(\vec{H}_0\right)_\rho\Big|_{surface} = \rho^\gamma A \tag{11b}$$

$\gamma + 1$:

$$E_1 = \partial\left(\vec{H}_0\right)_y/\partial x - \partial\left(\vec{H}_0\right)_x/\partial y \tag{12a}$$

$$k\left(\vec{H}_0\right)_x = \partial E_1/\partial y; \quad k\left(\vec{H}_0\right)_y = -\partial E_1/\partial x. \tag{12b}$$

Equation (11a) indicates that the electric field is zero everywhere inside the perfect conductor including the surface. Therefore, the expansions in (10) have the same physical meaning as in the classical surface impedance theory for smooth surfaces in spite of the fact that in the latter case $\gamma = 0$ in the power series. To obtain $E_1$ using (12a), we have to know the normal derivatives of the functions $(\vec{H}_0)_x$, $(\vec{H}_0)_y$ along the conductor's surface. For this purpose the following boundary value problem should be considered.

## IV. Electric and Magnetic Fields in the Conductor Near the Corner

Distribution of the magnetic field inside a conductor can be described using the following diffusion equations:

$$\Delta H_x = j\omega\sigma\mu H_x; \quad \Delta H_y = j\omega\sigma\mu H_y \tag{13}$$

where $\Delta$ denotes the Laplace operator. By rewriting (13) in terms of nondimensional variables introduced in the previous section and substituting the expansions (10), the following equations are obtained:

$$\Delta(\vec{H}_0)_x = k^2(\vec{H}_0)_x; \tag{14a}$$

$$\Delta(\vec{H}_0)_y = k^2(\vec{H}_0)_y; \qquad k^2 = 2j. \tag{14b}$$

The problems in (14) can be supplied by the following boundary conditions based on (11b):

$$\left.\begin{array}{ll} \alpha = 2\pi\colon \left(\vec{H}_0\right)_x = \rho^\gamma A; & \alpha = 3\pi/2\colon \left(\vec{H}_0\right)_x = 0 \\ \rho = 0\colon \left(\vec{H}_0\right)_x \to \infty; & \rho \to \infty\colon \left(\vec{H}_0\right)_x \to 0 \end{array}\right\} \tag{15a}$$

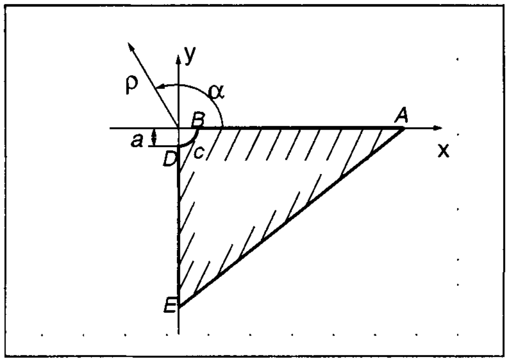


Fig. 2. Boundary value problem (16).

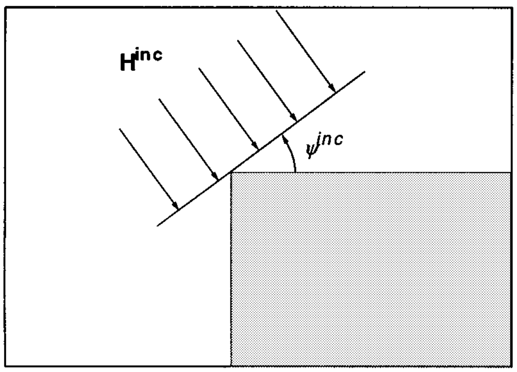


Fig. 3. Diffraction of the plane wave by conducting edge.

$$\left.\begin{array}{ll}\alpha = 2\pi\colon \left(\vec{H}_0\right)_y = 0; & \alpha = 3\pi 2\colon \left(\vec{H}_0\right)_y = \rho^{\gamma} A \\ \rho = 0\colon \left(\vec{H}_0\right)_y \to \infty; & \rho \to \infty\colon \left(\vec{H}_0\right)_y \to 0\end{array}\right\}. \tag{15b}$$

In (15) we used the fact that the normal magnetic field vanishes on the surface of a perfect conductor. Note that formulations (14), (15) can give only an approximate solution of the problem of the magnetic field diffusion inside the real conductor near the edge because the boundary conditions for the ideal conductor are used. It means that terms containing $\vec{H}_1$ have been neglected, but it is a suitable approximation for the calculation of $E_1$ as follows from (11), (12).

Formulation (14a), (15a) is analogous to (14b), (15b), so we can restrict ourselves to consideration of the problem for $(\vec{H}_0)_x$. Let $f(\alpha)$ be any integrable function. Then it can be proven [9] that the solution of (14a), (15a) can be found as a limit of the solution of the following boundary value problem in the domain $ABcDE$ (Fig. 2):

$$\begin{aligned}&\frac{\partial^2(\vec{H}_0)_x}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial(\vec{H}_0)_x}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2(\vec{H}_0)_x}{\partial\alpha^2} - k^2(\vec{H}_0)_x = 0\\ &(\vec{H}_0)_x\Big|_{\alpha=2\pi} = A\rho^{\gamma}; \quad (\vec{H}_0)_x\Big|_{\alpha=3/2\pi} = 0;\\ &(\vec{H}_0)_x\Big|_{\rho=a} = f(\alpha); \quad (\vec{H}_0)_x\Big|_{\rho=\infty} = 0;\end{aligned} \tag{16}$$

as the radius $a$ of arc $BcD$ tends to zero. The problem (16) has been solved using the Green function technique and the solution can be represented in the form:

$$\begin{aligned}(\vec{H}_0)_x &= T_1 + T_2 \qquad (17)\\ T_1 &= \kappa \sum_{n=1}^{\infty} \frac{\mathrm{K}_{\lambda_n}(\kappa\chi)}{\mathrm{K}_{\lambda_n}(\kappa)} F_n \sin(2\alpha n/3);\\ T_2 &= A\kappa e^{-i\pi/2} \sum_{n=1}^{\infty} \frac{\mathrm{K}_{\lambda_n}(\kappa\chi)}{\mathrm{K}_{\lambda_n}(\kappa)} \lambda_n \sin(2\alpha n/3)\\ &\quad\times \int_1^{\chi} \tilde{\chi}^{\gamma} \left(I_{\lambda_n}(\kappa\tilde{\chi})\mathrm{K}_{\lambda_n}(\kappa) - I_{\lambda_n}(\kappa)\mathrm{K}_{\lambda_n}(\kappa\tilde{\chi})\right) d\tilde{\chi}\\ &\quad + A\kappa i \sum_{n=1}^{\infty} \lambda_n \sin(2\alpha n/3)\\ &\quad\times \left(I_{\lambda_n}(\kappa\chi) - \frac{I_{\lambda_n}(\kappa)}{\mathrm{K}_{\lambda_n}(\kappa)}\mathrm{K}_{\lambda_n}(\kappa\chi)\right)\\ &\quad\times \int_{\chi}^{\infty} \tilde{\chi}^{\gamma}\mathrm{K}_{\lambda_n}(\kappa\tilde{\chi})\, d\tilde{\chi};\\ &\lambda_n = 2n/3; \quad \chi = \rho/a; \quad \kappa = ka\end{aligned}$$

where $F_n$ is the Fourier transform of the function $f(\alpha)$; $I_m$ and $K_m$ are the $m$-order modified Bessel and Hankel functions, respectively. It can be shown that $T_1$ vanishes when $a \to 0$ and (17) takes the form:

$$\begin{aligned}(\vec{H}_0)_x\Big|_{a\to 0} &= \sum_{n=1}^{\infty} \frac{2n}{3} e^{-i\pi/2} \sin(2n\alpha/3)\\ &\quad\times A\left\{\mathrm{K}_{\lambda_n}(k\rho)\int_0^{\rho} \tilde{\rho}^{\gamma} I_{\lambda_n}(k\tilde{\rho})\, d\tilde{\rho}\right.\\ &\quad\left. + I_{\lambda_n}(k\rho)\int_{\rho}^{\infty} \tilde{\rho}^{\gamma}\mathrm{K}_{\lambda_n}(k\tilde{\rho})\, d\tilde{\rho}\right\}.\end{aligned} \tag{18}$$

Problem (14b), (15b) has been solved in the same way and the result can be written in the form:

$$\begin{aligned}(\vec{H}_0)_y\Big|_{a\to 0} &= \sum_{n=1}^{\infty} \frac{2n}{3} e^{-i\pi/2}(-1)^n \sin(2n\alpha/3)\\ &\quad\times A\left\{\mathrm{K}_{\lambda_n}(k\rho)\int_0^{\rho} \tilde{\rho}^{\gamma} I_{\lambda_n}(k\tilde{\rho})\, d\tilde{\rho}\right.\\ &\quad\left. + I_{\lambda_n}(k\rho)\int_{\rho}^{\infty} \tilde{\rho}^{\gamma}\mathrm{K}_{\lambda_n}(k\tilde{\rho})\, d\tilde{\rho}\right\}.\end{aligned} \tag{19}$$

Asymptotic analysis of (18), (19) yields distributions of the functions $(\vec{H}_0)_x$ and $(\vec{H}_0)_y$ in the conductor near the corner:

$$(\vec{H}_0)_x = A\left(\frac{2}{k\rho}\right)^{1/3} \frac{\cos(\alpha/3)}{\cos(2\pi/3)}, \tag{20a}$$

$$(\vec{H}_0)_y = A\left(\frac{2}{k\rho}\right)^{1/3} \frac{\sin((2\pi-\alpha)/3)}{\cos(2\pi/3)}. \tag{20b}$$

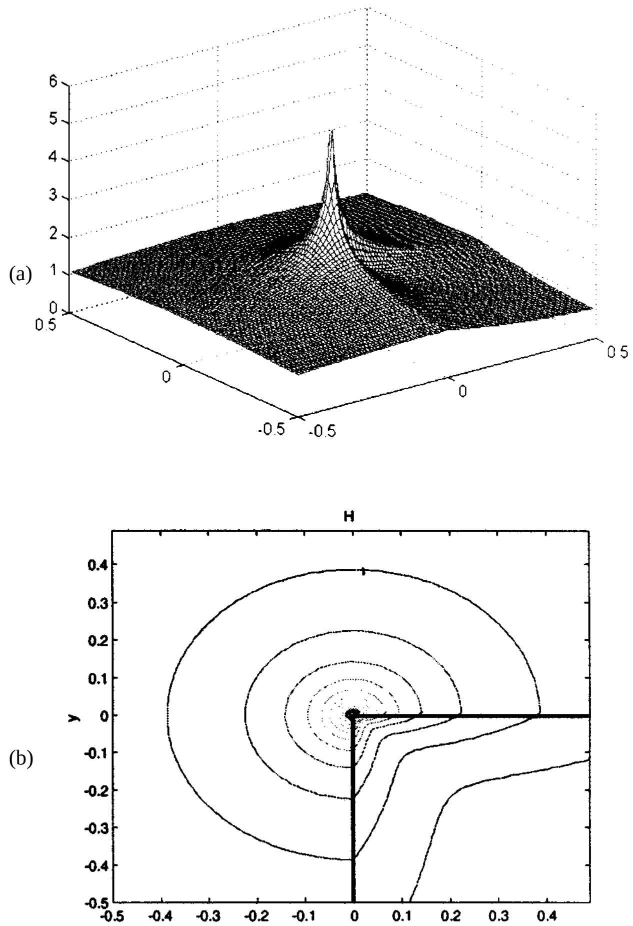


Fig. 4. (a) The magnetic field intensity near the corner. (b) The magnetic field intensity near the corner.

Substituting (20) into (12a), the electric field is obtained:

$$E_1 = \frac{2}{3} A\rho^{-4/3} \left[ \frac{2}{3} \sin\alpha \cos\frac{\alpha}{3} + \frac{2}{3} \sin\frac{2\pi - \alpha}{3} \cos\alpha + \frac{1}{3} \sin\frac{4\alpha}{3} + \frac{1}{3} \sin\frac{2\pi - 4\alpha}{3} \right]. \quad (21)$$

To illustrate the theory, consider the problem of diffraction of a plane wave by an imperfectly conducting edge as shown in Fig. 2. The solution of the problem for the perfect conductor is known [3] and can be represented in the form:

$$H_x|_{\rho \ll 1} \approx \frac{\sqrt{3\pi}k}{\Gamma(2/3)} H^{inc} e^{i\pi/3} \left(\frac{2}{k\rho}\right)^{1/3} \sin\frac{2\psi^{inc}}{2} \cos\frac{\alpha}{3}, \quad (22a)$$

$$H_y|_{\rho \ll 1} \approx \frac{\sqrt{3\pi}k}{\Gamma(2/3)} H^{inc} e^{i\pi/3} \left(\frac{2}{k\rho}\right)^{1/3} \sin\frac{2\psi^{inc}}{2} \sin\frac{\alpha}{3}. \quad (22b)$$

Thus in this case the coefficient $A$ takes the form

$$A = H^{inc} e^{i\pi/3} \frac{\sqrt{3\pi}k}{\Gamma(2/3)} \sin(2\psi^{inc}/3).$$

The distribution of the magnetic field intensity in the conducting and dielectric regions near the edge was calculated using (19), (21) and is shown in Fig. 4(a) and (b).

## V. Calculation of the Surface Impedance

By switching in (5) to nondimensional variables and substituting series (10), the following representation of the surface impedance is obtained:

$$Z_i = \frac{E}{H_i} = \frac{pE_1 + O(p^2)}{(\vec{H}_0)_i + p(\vec{H}_1)_i + O(p^2)} = \frac{pE_1}{(\vec{H}_0)_i} + O(p^2)$$
$$i = x, y. \quad (23)$$

Substitution of (20), (21) into (23) leads to the following expressions:

$$Z_x = -p\frac{\tan(\pi/6)}{\rho}; \quad Z_y = p\frac{\tan(\pi/6)}{\rho}. \quad (24)$$

Returning in (24) to dimensional variables, we finally obtain:

$$Z_x = -\frac{\tan(\pi/6)}{\sigma\rho}; \quad Z_y = \frac{\tan(\pi/6)}{\sigma\rho}. \quad (25)$$

## References

[1] E. M. Deeley, "Surface impedance near edges and corners in three dimensional media," *IEEE Trans. Magn.*, vol. 26, no. 2, pp. 712–714, 1990.

[2] W. Jingguo and J. D. Lavers, "Modified surface impedance boundary conditions for 3D eddy current problems," *IEEE Trans. Magn.*, vol. 29, no. 2, pp. 1826–1829, 1993.

[3] G. A. Greenberg, "On a method of solving the fundamental problem of electrostatics and allied problems" (in Russian), *Zh. Exp. Teor. Phyz.*, pt. I, vol. 8, no. 3, pp. 221–252, March 1938.

[4] J. Meixner, "The behavior of electromagnetic fields at edges," *IEEE Trans. Antennas Prop.*, vol. 20, no. 4, pp. 442–446, 1972.

[5] V. Bladel, *Electromagnetic Fields*, 1985, A SUMMA Book, ch. 5 and 12.

[6] K. Nikoskinen and I. Lindell, "Image solution for Poisson's equation in wedge geometry," *IEEE Trans. Antennas Prop.*, vol. 43, no. 20, pp. 179–187, 1995.

[7] S. Yuferev and L. Kettunen, "A unified surface impedance concept for both linear and nonlinear skin effect problems," *IEEE Trans. Magn.*, vol. 35, no. 3, pp. 1454–1457, 1999.

[8] S. Yuferev and N. Ida, "Selection of the surface impedance boundary conditions for a given problem," *IEEE Trans. Magn.*, vol. 35, no. 3, pp. 1486–1489, 1999.

[9] G. A. Greenberg, *Selected Topics in the Mathematical Theory of Electric and Magnetic Phenomena* (in Russian). Moscow: Academy Sci USSR, 1948.

[10] ——, "On a method of solving the fundamental problem of electrostatics and allied problems" (in Russian), *Zh. Exp. Teor. Phyz.*, pt. II, vol. 9, no. 6, pp. 725–728, June 1939.