'A Posteriori' Element by Element Local Error Estimation Technique and 2D & 3D Adaptive Finite Element Mesh Refinement

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Abstract **– A new approach for estimating an *'a posteriori'* error locally on an element by element basis for the adaptive refinement of a class of 2D and 3D boundary value problems has been investigated in this paper. It is necessary to have an efficient, robust and reliable error estimator to generate an optimal adaptive mesh. In this work an *'a posteriori'* error is computed by solving a local problem on a patch of elements. It is simple to implement and is computationally inexpensive. This method computes the local as well as global error by using a *h*–version of adaption with quadratic shape functions to solve the local problem. The refinement algorithm makes use of a minimal hierarchical tree based data structure which minimizes the amount of tree traversal normally required during the refinement process. The efficiency of the local error estimation technique has been demonstrated by the adaptive refinement of an ac transmission line problem in 2D and an eddy current problem employing complex magnetic vector potential formulation in 3D. The coarse mesh and the refined optimal meshes and the numerical results substantiate that the local *'a posteriori'* error estimate is efficient and simple to use for most practical applications.**

I. INTRODUCTION

Numerical modeling procedures like the Finite Element (FE) method introduces modeling errors due to the insufficient discretization of the continuum. The error may range from 5 to 15% in various engineering applications. Some scientific and engineering applications require solutions with a very high accuracy. The discretization error in the problem domain can be minimized by identifying and refining the critical regions of the domain where the most error occurs. It is achieved by computing an *'a posteriori'* error from the one time solution. The error indicator and error estimators provide a reliable way of steering the mesh refinement process. An efficient adaptive refinement relies mainly on the appropriate choice of an error estimation strategy and a suitable mesh refinement algorithm to generate an optimal mesh to enhance the accuracy level of the solution. A reasonably good *'a posteriori'* error estimate not only identifies the critical regions of the problem domain and steers the adaptive mesh refinement, but also allows assessment of the quality of the computed solution. The error estimates may be broadly classified as *'a posteriori'* and *'a priori'*. Due to the insufficient *'a priori'* data relating to the nature and physical behavior of a problem and its solution, it is very difficult to estimate the quantity and quality of error *'a priori'* even before the actual solution is computed. Hence most of the practically usable error estimates are computed *'a posteriori'*. In this investigation a simple method for *'a posteriori'* error estimate using a local error problem is reported. The error estimate was tested by applying it to a class of 2D and 3D eddy current problems. This paper focuses on the effectiveness of the local error estimation strategy and also the mesh refinement algorithm for the generation of optimal (or nearly optimal) meshes for 2D and 3D problems.

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II. AN *'A POSTERIORI'* ERROR ESTIMATION

To effectively control the *'discretization'* error an *'a posteriori'* error estimator should be robust, reliable and accurate and must control the process of adaptively refining the mesh in an economical way. Many heuristic error estimation strategies with mathematical analysis and extensive results have been reported in the literature [1]–[8]. These strategies are developed either by utilizing the solution, its residual, or its gradient, or the post–processed form of the solution or the system energy. The mesh optimization and quality assessment of the solution for practical applications mainly depend on the choice of an appropriate error estimate. The reliability of an error estimate should ensure proper adaptation on all classes of problems independent of the nature of problems and also the material types (either isotropic or anisotropic) involved in the problem domain.

A. Local Error Problem Method of 'a posteriori' Error Estimation

Most of the problems in electrostatic and electromagnetic fields possess a complexity in geometry and multiplicity in material interfaces. It is required to have a very reliable and computationally efficient error estimate. The choice of a local error estimate allows solution of a smaller problem at a subdomain level. The local error estimation utilizing the field quantity, its gradient or the energy of the system for electromagnetic problems has been reported in [2]–[4], [5]. A local error estimate can be based on the different forms of the field quantities and their related properties. A local error estimate can also utilize the dual energy concept to compute the lower and upper bounds of energy at subdomain level or the boundary discontinuity of an electromagnetic field components can be exploited to estimate error locally at an element level [7]–[8]. For example the discontinuity of the tangential component of magnetic flux density B_t and the discontinuity in the normal component of the magnetic field intensity H_n and also the Maxwell stress tensor can be used as a measure of error.

The algorithm adopted for a local *'a posteriori'* error estimation is simple and computationally inexpensive. At each stage of refinement a local problem is constructed using the previous mesh and the global solution. The local error problem is a subdomain consisting of a patch of elements connected to

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a regular node. Using the global solution, appropriate Neumann or Dirichlet boundary conditions are imposed on the local problem. The local problem thus constructed has only a few nodes and hence is computationally less expensive. The local error problem $L\Phi = f$ on the subdomain is solved by using a quadratic interpolation polynomial with h-refinement. While solving the local problem, the boundary nodes and the constrained nodes due to the imposition of *'one–level'* rule for h-adaptive refinement are treated by a special procedure. By repeating this technique for all the active nodes in the entire problem domain, an improved solution is computed for each of the active nodes.

B. Error Measures

The optimal solution is obtained when the system energy converges in some norm, and so the energy norm as a measure of error is natural and exact. The error in the L_2 energy norm is computed using the two sets of solutions available on each element. Let Ω be the domain and Φ_{ex} , Φ are the exact and computed solutions respectively, then the error in L_2 energy norm is,

$$\| e \|_{L_2} = \left\{ \int_\Omega e^T e \, d\Omega \right\}^{\frac{1}{2}} \quad where \quad e = \Phi_{ex} - \Phi \tag{1}$$

'Root Mean Square' energy norm of local error on each element is,

$$\| \Delta e \| = \left\{ \| e \|_{L_2}^2 \; / \; \Omega \right\}^{\frac{1}{2}} \tag{2}$$

The global energy norm error is,

$$\| e \|^2 = \sum_{i=1}^{i=n} \left\{ \| e \|_{L_2}^2 \right\}_i \tag{3}$$

The relative energy norm error in percentage using exact solution Φ_{ex} is,

$$\eta = \left\{ \| e \| \; / \; \| \Phi_{ex} \| \right\} * 100\% \tag{4}$$

For most practical problems, the exact solution may not be available and so the exact solution is approximated by adding the error component with the approximated FE solution to compute the global relative error. The global relative error in terms of the approximated solution is,

$$\eta = \left\{ \| e \| \; / \; \left(\| \Phi \|^2 + \| e \|^2 \right)^{\frac{1}{2}} \right\} * 100\% \tag{5}$$

Where

$$\| \Phi \|^2 = \sum_{i=1}^{i=n} \| \Phi \|_i^2 \tag{6}$$

The admissible error in the energy norm in an element is derived using the global relative error and the number of elements,

$$\| e \|_a = \frac{\eta}{100} \left\{ \frac{\| \Phi \|^2 + \| e \|^2}{N} \right\}^{\frac{1}{2}} \tag{7}$$

A reliable error criterion for elementwise refinement can be derived using the admissible error in energy norm $\| e \|_a$ and the local error indicator $\| e \|_i$ as follows,

$$\zeta_i = \| e \|_i / \| e \|_a \tag{8}$$

By using the local error indicator and refinement ratio ζ_i an element gets refined whenever,

$$\zeta_i > 1 \qquad \forall \; \| e \|_i > \| e \|_a \tag{9}$$

The algorithm for the local error problem employs a h–type refinement procedure along with quadratic shape functions. The local error in L_2 energy norm computed on each element is used to identify the critical regions of the problem domain by comparing it with the allowable error $\| e \|_a$. The global relative error acts as a stopping criterion determining the number of refinements necessary to generate an optimal mesh. The equi–distribution of error occurs over the domain when the error indicators are asymptotically equal and the mesh becomes optimal (or nearly optimal).

III. ADAPTIVE MESH REFINEMENT

It is necessary to have an efficient algorithm for adaptive mesh refinement and a data structure to perform the various data management functions in the refinement process. Many adaptive refinement schemes and the associated data structures have been developed for practical adaptive FE computation over the years [9]. Based on the type of refinement policy adopted as to whether the solution is improved by reducing the size of the element ($h_{max} \rightarrow 0$) or increasing the order of the approximating polynomial ($p \rightarrow \infty$) or combining both, they are classified as h, p and h–p versions. In this investigation, the global mesh refinement is implemented using a h–refinement policy. The mesh refinement algorithm imposes a *'one–level'* rule by which no two neighboring elements can have a refinement level difference of more than one [10]. The *'one–level'* rule facilitates generation of a graded and smooth mesh. The refinement generates four equal quartets in 2D and eight congruent octets in 3D. The h–refinement data base makes use of a minimal hierarchical tree based data structure which minimizes the amount of tree travel normally required during the refinement process. The refinement algorithm employs first order isoparametric quadrilateral in 2D and hexahedral brick elements in 3D.

IV. NUMERICAL TEST RESULTS AND PERFORMANCE ANALYSIS

The local error problem method of *'a posteriori'* error estimate and the mesh refinement algorithm have been tested by applying them to solve linear self adjoint boundary value problems in 2D and 3D. In 2D, an electromagnetic eddy current problem concerning an infinite parallel transmission line transmitting an AC current at a frequency of *120Hz* is adaptively solved. The transmission line is located *1cm* above a thick conducting plate with a thickness of *1cm*. The problem is formulated in terms of a complex magnetic vector potential A.

The following is the Poisson equation used to compute the electromagnetic vector potential A.

$$-\frac{1}{\mu}\left(\nabla x \nabla x A\right) = J_s - j\omega\sigma A \qquad (10)$$

The magnetic vector potential A is used as the primary field quantity to compute the local error in this problem. The local problem has been formulated and the local error estimate is applied on the local subdomain constructed after the initial solution. Using the quadratic approximation polynomial and the h-refinement, the solution at each of the active nodes is improved and the local error is computed and used for the global mesh refinement. The intermediate coarse mesh and the final adaptive meshes are shown in fig. 1. The corresponding equi-potential plots are shown in fig. 2.

In order to evaluate the performance of the local error estimation, a 3D electromagnetic field eddy current problem employing a complex magnetic vector potential A has been adaptively solved. The problem is to compute the eddy current losses at *60Hz* in a finite length aluminum bar having a square cross section surrounded by an exciting coil. The detailed analysis using the complex magnetic vector potential A based on the minimization of energy functional along with experimental results for this problem is provided in [11]–[12]. An eddy current constraint formulation computing an improved solution is investigated in [13]. The aluminum bar is *2.54x2.54 cm* cross section with a length of *20.32 cm* and the excitation coil is of length *8.128 cm* and the thickness of the coil is *0.685 cm*. The problem domain is shown in fig. 3. Utilizing the symmetry of the domain geometry, only one octant of the aluminum bar and the coil need to be modeled. The first order isoparametric hexahedral brick elements are used to discretize the problem domain. The coarse mesh and the adaptive meshes are shown in fig. 4. The local error estimate was applied and the induced eddy current loss in the conducting aluminum bar is computed adaptively for different sets of source currents of *1.0 A*, *1.5A* and *2.0 A*. The power loss computed by adaptive mesh refinement is tabulated along with the corresponding experimental values for comparison in table(1). The convergence of error in the computed eddy current power loss for different sets of source current is shown in the error convergence plot in fig. 5.

In the 2D eddy current problem, as evidenced from the nature of the field, it will be stronger surrounding the transmission line bars and so there will be a higher mesh density compared to other regions of the domain. Accordingly the refinement of the mesh concentrates around the transmission line bar as shown in fig. 1. It is again verified by the improved solution plot as shown by the equi-potential lines in fig. 2. In the case of a 3D electromagnetic eddy current problem, as shown by the adaptive mesh in fig. 4, more error in the field quantity A is introduced in the vicinity of the material interface region especially between the conductor and the coil. Due to this discontinuity, more elements got refined in the material interface region where the field is stronger compared to other parts of the problem domain. The power loss due to the induced eddy current in the conducting aluminum bar is improved from one refinement to the other. This is verified by the sequence of adaptive meshes and the error convergence plot. Thus the numerical test results and the adaptive meshes and also the convergence plots prove the effectiveness of the proposed *'a posteriori'* local element by element error estimation technique.

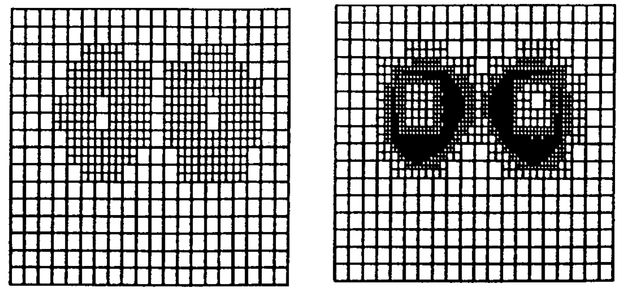

Fig. 1 : Intermediate Mesh and Refined Final Mesh

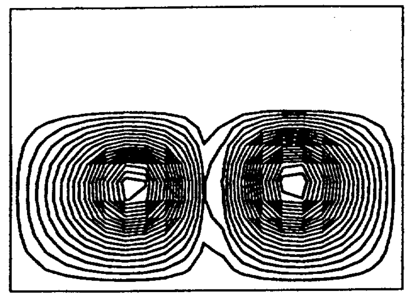

Fig. 2a : Equi–Potential Plot of an Intermediate Mesh

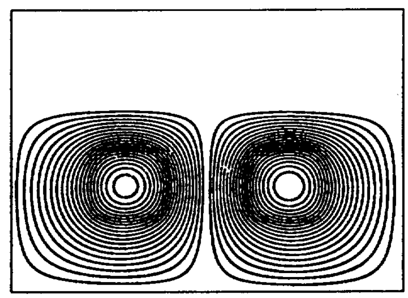

Fig. 2b : Equi–Potential Plot of a Final Mesh

TABLE–1

Comparison of eddy Current Power Loss with Experimental Values

Excitation Current in Amps	Experimental Values in Watts	Power Loss By Adaptive FE Method	Error in %
1.0	0.58	0.5645	2.67
1.5	1.56	1.508	3.33
2.0	3.52	3.3728	4.18

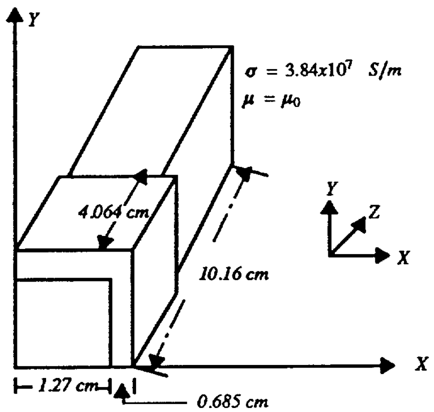

Fig. 3 : Aluminum Conducting Bar and the Excitation Coil

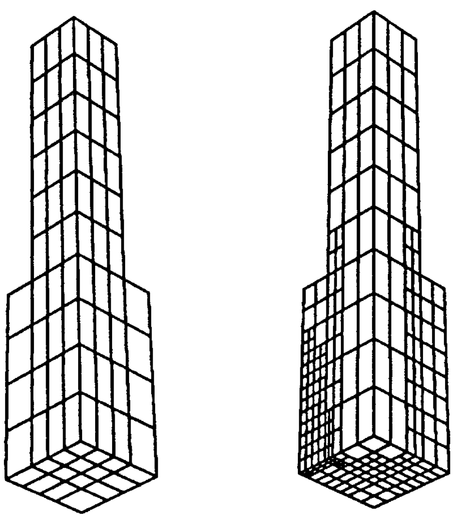

Fig. 4 : Initial and Refined 3D Meshes (Material Regions Only)

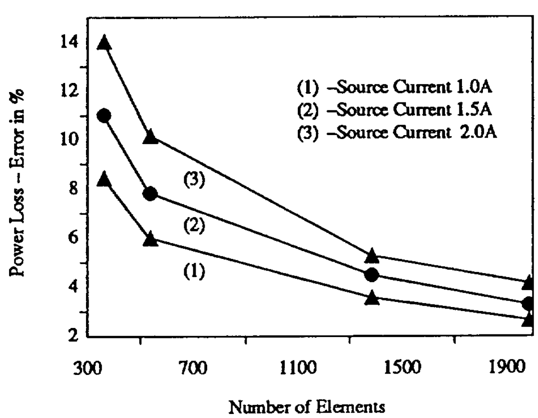

Fig. 5 : Adaptive Error Convergence Plot

V. CONCLUSION

A local problem method of element by element *'a posteriori'* error estimation technique and an adaptive finite element mesh refinement for 2D and 3D problems have been presented. The local error estimation has been applied to adaptively solve eddy current problems employing complex magnetic vector potential formulation in 2D and 3D. The sequence of adaptive meshes and the corresponding solution plots show the effectiveness of the proposed local error estimate. The improved power loss due to induced eddy currents is computed from the numerical experiments and compared with experimental data. From the numerical experiment, it is verified that the error estimate is computationally inexpensive, and is effective in computing the local and global errors. The performance evaluation and the practical application of the local element by element error estimation method is established from the adaptive meshes and the numerical test results of the eddy current problems.

REFERENCES

[1] A.R. Pinchuk & P.P. Silvester, 'Error Estimation for automatic adaptive Finite Element Mesh Generation', *IEEE Trans. on Magnetics*, Vol .21, No.6, pp. 2551–2554 (1985).

[2] P. Fernandes, P. Girdinio, P. Molfino, G. Molinari & M. Repetto, 'A comparison of adaptive strategies for mesh refinement based on *A posteriori* local error estimation procedures ', *IEEE Trans. on Magnetics*, Vol. 26, pp. 795–798 (1990).

[3] P. Fernandes, P. Girdinio, P. Molfino & M. Repetto, 'Local error estimates for adaptive mesh refinement', *Int. J. Numer. Methods. Eng.*, Vol. 24, pp. 299–302 (1988).

[4] K. Adamiak, 'Local Error Indicator in Finite Element Analysis of Laplacian Fields Based on the Green Integration Formula', *Int. J. Numer. Methods. Eng.*, Vol. 33, pp. 1625–1642 (1992).

[5] S. Biddlecombe, J. Simkin & C.W. Trowbridge, 'Error Analysis in Finite Element Models of Electromagnetic Fields', *IEEE Trans. on Magnetics*, Vol. MAG–22, No. 5, pp. 811–813 (1986).

[6] Z.J. Cendes & D.N. Shenton, 'Adaptive Mesh Refinement in the Finite Element Computation of Magnetic Fields', *IEEE Trans. on Magnetics*, Vol. 21, pp. 1811–1815 (1985).

[7] J. Penman & J.R. Fraser, 'Dual and Complementary Energy methods in Electromagnetism', *IEEE Trans. on Magnetics*, Vol. 19, No. 6, pp. 2311–2316 (1983).

[8] M.G. Vanti, A. Raizer & J.P.A. Bastos, 'A Magnetostatic 2D Comparison of Local Error Estimators in FEM', *IEEE Trans. on Magnetics*, Vol. 29, No. 2, pp. 1902–1905 (1993).

[9] G.F. Carey, M. Sharma, K.C. Wang, 'A class of Data Structures for 2D and 3D Adaptive Mesh Refinement ', *Int. J. Numer. Methods. Eng.*, Vol. 26, pp. 2607–2622 (1988).

[10] K.C. Chellamuthu & Nathan Ida, 'Algorithms and data structures for 2D and 3D Adaptive Finite Element mesh Refinement', unpublished.

[11] N.A. Demerdash, O.A. Mohammed, T.W. Nehl, F.A. Fouad, R.H. Miller, 'Solution of eddy current Problems using Three Dimensional Finite Element Complex Magnetic Vector Potential', *IEEE Trans. on Power Apparatus and Systems*, Vol. PAS–101, No. 11, pp. 4222–4229 (1982).

[12] N. A. Demerdash, T.W. Nehl, F.A. Fouad, O. A. Mohammed, 'Three Dimensional Finite Element Vector Potential Formulation of Magnetic Fields in Electrical Apparatus', *IEEE Transactions on Power Apparatus and Systems*, Vol. PAS–100, No. 8, pp. 4104–4111 (1981).

[13] Hong Song & Nathan Ida, 'An eddy current constraint formulation for 3D electromagnetic field calculations', *IEEE Trans. on Magnetics*, Vol. 27, No. 5, pp. 4012–4015 (1991).