Reliability Assessment of an '*a posteriori*' Error Estimate for Adaptive Computation of Electromagnetic Field Problems

K.C. Chellamuthu and Nathan Ida

Department of Electrical Engineering
The University of Akron
Akron, OH – 44325 – 3904

Abstract – **Optimal performance of an adaptive Finite Element (FE) computation depends on the availability of a reliable and computationally efficient '*a posteriori*' error estimation strategy. The reliability of an error estimate ensures that the quality of the computed solution remains within a specified accuracy and also guarantees that the error estimate applies uniformly over the entire problem domain. Reliability analysis of two different error estimates with a model problem and numerical test results are reported in this paper. A mathematical model for the reliability assessment of an '*a posteriori*' error estimate through asymptotic exactness is also presented. The reliability of the performance of two different error estimates is assessed by adaptively solving a linear boundary value problem.**

I. INTRODUCTION

In finite element analysis, discretization error plays a major role in determining the final solution accuracy. Solution accuracy in the range of 5%–15% is acceptable for most engineering applications but for some applications very high accuracy in the range of 2%–3% is required. In order to improve the accuracy of the solution adaptively, an error estimate should be reliable enough to identify the critical regions of the domain which has larger errors. It requires an efficient and quantitative error estimate to accurately gauge the error in the solution. Thus the efficiency of an adaptive FE computation depends on the availability of a computationally robust and reliably stable error estimate. The reliability of an error estimate provides a measure of accuracy of the computed solution. Reliability analysis for electromagnetic field problems using two different '*a posteriori*' error estimates is proposed in this paper. A mathematical model with a problem definition for reliability analysis is presented in the first part. In the second part, the theory of reliability assessment of an error estimate through asymptotic exactness is outlined. Numerical test results for a sequence of nearly optimal adaptive meshes for a 2D problem is presented in the third part of the paper.

II. RELIABILITY OF AN '*A POSTERIORI*' ERROR ESTIMATE IN ADAPTIVE FINITE ELEMENT COMPUTATION

For adaptive solution, many '*a posteriori*' error estimation strategies are available [1–3]. Often the error estimates are based on the weak variational formulation incorporated in the FE problem definition. Irrespective of the field variable used and the methodology employed, an error estimate should perform uniformly and reliably at all stages of a computation so as to make an adaptive process more effective. In addition to reliability and robustness, an error estimate should be capable of effectively handling geometric and other singularities. It should also gauge the error accurately with different physical modeling characteristics. An '*a posteriori*' error estimate helps to determine the optimality of the mesh when error becomes equal on all the elements of the domain. Thus an asymptotically optimal mesh is obtained when all the error indicators attain asymptotical exactness.

Manuscript received July 6, 1994.

A. *Reliability Analysis*

Reliability is often based on an '*a priori*' mathematical analysis and also on an '*a posteriori*' quantitative error estimation. Due to uncertainty in the input data '*a priori*' mathematical analysis is of little use for practical error estimation. Often '*a posteriori*' error estimates are used for estimating reliable error bounds utilizing the data available during the process itself. With the exception of a few methods [4–6,8–10] most '*a posteriori*' error estimation strategies are based on heuristic reasoning without extensive mathematical treatment for reliability analysis. Error estimates are often based on benchmark computations satisfying specific computational goals. This is due to the fact that the quality of error estimates are sensitive to the complexity and structure of a problem domain, the mesh quality and the nature of singularities.

B. *Mathematical Model for Reliability Analysis*

The rate of convergence of an approximate solution corresponding to the sequence of adaptive meshes is governed by the degree of deviation of a mathematical model from the exact physical characteristics of the problem. The rate of convergence is often characterized by the regularity of the solution, the singular behavior of the problem and the reliability of the mathematical model adopted. The reliability of an error estimate is judged by the asymptotic exactness which is based on the convergence properties of the problem. For a well–posed problem satisfying the consistency and compatibility conditions, the reliability of an error estimate can be best measured by means of local and global effectivity indices. From the convergence characteristics of the finite element formulation, the sequence of approximate solutions in an adaptive process ultimately converges to exact solutions in energy norm satisfying the given error tolerance. If the solution converges to the exact solution in an optimal mesh, the error in the solution also attains asymptotic convergence. The well–posedness of a problem involving the existence and uniqueness of an exact and approximate solution and the guarantee of convergence helps to reliably estimate the error in

0018-9464/95$04.00

the solution. It leads to the application of a rigorous mathematical and functional analysis to the error estimation.

All elliptic boundary value problems possessing the property of linear self adjoint positive definiteness should satisfy the set of consistency conditions such as well–posedness, continuity and completeness in order to achieve an asymptotic convergence of a solution.

C. *A Model Boundary Value Problem*

In order to analyze the consistency conditions for asymptotic convergence, we define a model boundary value problem as follows: Consider we seek to develop a model (weak Galerkin formulation) for approximation of a problem by an adaptive FE method. Let $\Omega \subset R^2$ be a smooth bounded domain with a boundary $\Gamma = d\Omega = \Gamma_\phi + \Gamma_a$ consisting of two disjoint sections Γ_ϕ and Γ_a such that $\Gamma_\phi \cap \Gamma_a = \phi$ on which we wish to find the solution Φ, then,

$$-\Delta\Phi = f \quad in \quad \Omega \tag{1}$$

$$\Phi = \Phi_0 \quad on \quad \Gamma_\phi \text{ and } \frac{\partial\Phi}{\partial n} = g \quad on \quad \Gamma_a \tag{2}$$

To compute a reliable *'a posteriori'* error estimate for the above problem, it should satisfy the conditions such as well–posedness of a problem (existence and uniqueness property), coercivity of the bilinear form or the V –Ellipticity condition and symmetry of spaces.

Assuming that the solution Φ lies in the standard Sobolev space $H^1(\Omega)$ of all functions which together with their generalized first derivatives are square integrable on Ω, then the weak formulation can be defined using the following abstract mathematical forms,

$$V(\Omega) = \{v \in H^1(\Omega) : v = 0 \quad on \quad \Gamma_\phi\} \tag{3}$$

Find:

$$\Phi \in \Phi_0 + V \text{ such that } B(\Phi, v) = L(v) = (f, v) \ \forall v \in V, \tag{4}$$

where $B(.,.)$ and $L(.)$ are bilinear and linear forms defined on the finite dimensional approximation space V as follows,

$$B : VxV \rightarrow R \ : B(\Phi, v) = \int_\Omega (\nabla\Phi . \nabla v) \ d\Omega \tag{5}$$

$$L : V \rightarrow R \text{ and } L(v) = \int_\Omega fv \ d\Omega + \int_{\Gamma_a} gv \ ds \tag{6}$$

By using the classical projection theorem (the Lax–Milgram theorem) [7], the uniqueness and existence of a weak solution for linear elliptic boundary value problems satisfying the following conditions can be verified:

Let $\exists\alpha_1, \alpha_2, \alpha_3 > 0$ such that

$$a) \ B(\Phi,\Phi) \geq \alpha_1 \|\Phi\|^2 \text{ for } \forall\Phi \in V \quad Coercivity \tag{7}$$

$$b) \ B(\Phi,\Phi) \geq 0 \text{ for } \forall\Phi \in H^1(\Omega) \quad Positive\ definite \tag{8}$$

$$c) \ B(\Phi, v) \leq \alpha_2 \|\Phi\| \ \|v\| \ \forall\Phi, v \in V \quad Bilinear \tag{9}$$

$$d) \ \|L(\Phi)\| \leq \alpha_3 \|\Phi\| \text{ for } \forall\Phi \in V \quad Linear \tag{10}$$

The above analysis guarantees the existence and uniqueness of a weak solution to the problem and is used for computing the error. The basic consistency conditions such as well–posedness, compatibility conditions and operator characteristics such as the linearity, self adjointness and positive definiteness can be utilized to derive local and global error measures.

D. *Error Measures in Reliability Assessment*

A critical aspect of the reliability of an approximated solution is the availability of an accurate error measure. The accurate measure of an error in the solution provides a feasible and practical way of judging the degree of the quality of the solution. The error measure also gauges the size of an error present in the solution for adaptive accuracy improvement. The choice of an error measure depends on the goal of computation and the method employed. The convergence of solution to the exact solution occurs when the system energy converges in some energy norm. Hence the use of an energy norm as a measure of error is a natural choice. The energy norm is associated with the Sobolev space $H^1(\Omega)$. The computation of error measures based on the energy norm is employed in almost all *'a posteriori'* error estimation schemes [6]. In order to have a realistic understanding of the distribution of error and also to accurately assess the quality of the solution, different error measures are necessary. Among the error measures, energy norm, relative percentage energy norm error, local and global effectivity indices are commonly used for error analysis and mesh refinement.

The effectivity index can be used to predict the accuracy and distribution of error locally and globally over the solution domain. In an adaptive process when the solution converges, the estimated error also converges to the true error and so the effectivity index asymptotically converges to one. This convergence behavior is very useful to measure the reliability of an error estimate. The asymptotic exactness of an effectivity index allows one to specify reliable error bounds for its effective performance. A useful range of bounds is $0.8 \leq \Theta \leq 1.2$. In a practical application the approximate solution will never be exactly equal to the true solution but it will be within the specified error tolerance. If the value of the effectivity index is outside the above range, it can be concluded that either the error is overestimated or underestimated and hence the reliability of the method and also the quality of the solution will be poor for practical applications.

III. RELIABILITY ASSESSMENT OF AN ERROR ESTIMATE THROUGH ASYMPTOTIC EXACTNESS

The fact that an '*a posteriori*' error estimate is exact in the asymptotic rate of convergence in a global energy sense ensures that the solution satisfies the criteria of a convergence theorem. It establishes confidence in usage and reliability of the method employed. The asymptotic exactness verifies that the approximated solution over the domain is very close to the exact solution in the limit of the element size h_{max} tending to zero..

The effectiveness of an error estimation strategy can be gauged by the relation between the accuracy measured in the energy norm and the number of degrees of freedom [6]. An efficient and reliable error estimate will accurately measure the discretization error over a wide range of mesh spacings and on different classes of problems. In an adaptive FE system, the assessment of the quality of a system is relative to the criterion of optimality. The reliability of an '*a posteriori*' error estimate can be best measured by means of its effectivity index, $\Theta = \| e \| / \| e \|_{ex}$ where $\| e \|$ and $\| e \|_{ex}$ are computed and exact errors respectively in energy norm. For engineering applications with an accuracy range of (*say*, $\leq 10\%$) we require $| \Theta - 1.0 | \leq 0.2$. If $\Theta \geq 1.0$ the true error is overestimated rather than underestimated. Hence for practical applications the error estimate should be in an acceptable range. If $\Theta > 2.0$ *or* $\Theta < 0.5$ the error estimate will not be acceptable for practical applications. In order for an error estimate to be reliable and efficient, the range for an effectivity index should be $0.8 \leq \Theta < 1.2$. This result has been established through various numerical experiments for different practical problems [4–6].

The error estimator is said to be asymptotically exact if Θ converges to one whenever the true error in the solution converges to zero. From the effectivity index computed locally and globally, it is possible to conclude that the error estimate has an asymptotic rate of convergence *i.e*, $\Theta \rightarrow 1$ *as* $h_{max} \rightarrow 0$. In other words the sequence of approximate solutions over the adaptive meshes converges to exact solutions in the limit of maximum size of an element tending to zero or the degree of approximating polynomial tending to infinity. In a realistic sense h_{max} will never be zero and so the error estimate will be effective within an acceptable range depending on the kind of problem solved.

The effectivity index not only defines how reliably the error estimate performs, but also its asymptotic exactness [5,6,9]. This means that the estimated error tends to the exact value when the mesh is refined. It provides an efficient way of assessing the reliability of the method employed in the process. By establishing a mathematical relationship for proving the asymptotic exactness of the method, it is possible to verify and assess the reliability of the error estimation technique. The asymptotic exactness of the effectivity index Θ can be established as follows,

A. *Asymptotic Exactness of Effectivity Index*

Let Φ^* and $\| e^* \|$ be the post-processed solution and the error in energy norm ($\| e^* \| = \| \Phi_{ex} - \Phi^* \|$) corresponding to an intermediate refined mesh $M_i(\Omega)$, then the error estimate $\| e \|$ will be asymptotically exact if $\| e^* \| / \| e \|_{ex} \rightarrow 0$ as $\| e \|_{ex} \rightarrow 0$. To prove this we write the error estimate $\| e \|$ as follows,

$$\| e \| = \| \Phi^* - \Phi \| \equiv \| (\Phi_{ex} - \Phi) - (\Phi_{ex} - \Phi^*) \| \qquad (11)$$

Where Φ_{ex} and Φ are exact and current approximated solutions. Now by using the triangle inequality,

$$\| \Phi_{ex} - \Phi \| - \| \Phi_{ex} - \Phi^* \| \leq \| e \| \leq \| \Phi_{ex} - \Phi \| + \| \Phi_{ex} - \Phi^* \| \qquad (12)$$

$$\| e \|_{ex} - \| e^* \| \leq \| e \| \leq \| e \|_{ex} + \| e^* \| \qquad (13)$$

Dividing the equation (13) by $\| e \|_{ex}$ and using the effectivity index Θ the above equation can be rewritten as,

$$\left\{ 1 - \| e^* \| / \| e \|_{ex} \right\} \leq \Theta \leq \left\{ 1 + \| e^* \| / \| e \|_{ex} \right\} \qquad (14)$$

This shows that the effectivity index Θ is asymptotically exact and it approaches one as the ratio $\| e^* \| / \| e \|_{ex} \rightarrow 0$. By assuming the ratio $\alpha = \| e^* \| / \| e \|_{ex} = 0.2$, the asymptotic bound for the effectivity index is derived as:

$$(1 - \alpha) \leq \Theta \leq (1 + \alpha) \Rightarrow 0.8 \leq \Theta \leq 1.2 \qquad (15)$$

The effectivity index Θ in this range will provide a most reliable and efficient measure to estimate the error and also to assess the reliability of the method for most engineering applications.

IV. NUMERICAL TEST RESULTS AND RELIABILITY ASSESSMENT

The reliability of performance of two different '*a posteriori*' error estimates are assessed by applying them for adaptively solving a 2D electrostatic problem. The post-processing and the gradient of field method of error estimates are utilized to adaptively improve the solution of a 2D boundary value problem with an L-shaped domain with a corner singularity in the form of $r^{\frac{2}{3}} \sin \frac{2}{3} \theta$, where r and θ are polar coordinates [3]. The numerical test results for the two error estimates are shown in table-1 and table-2 respectively. The convergence of error in energy norm and the asymptotic convergence of the effectivity indices are illustrated in fig. 1 and fig. 2 respectively. From the numerical test results and the asymptotic convergence plot of effectivity indices, it is established that the reliability of performance of the proposed error estimates is within the useful range of bounds *i.e,* $0.8 \leq \Theta \leq 1.2$. From fig. 2 it is also observed that the degree of reliability of the gradient of field error estimate is superior to the post-processing error estimation scheme. This is demonstrated by the fact that the gradient of field error estimate provides an over estimate of the error in the solution whereas the post-processing method under estimates the error. From the numerical experiment, the asymptotic exactness of the effectivity index is verified and the reliability of the proposed error estimates is assessed.

TABLE – 1

Numerical Test results for Post-processing Error Estimate

Mesh Number	No. of Elements	Max Relative Error in %	Global Relative Error	Global Energy Norm ‖ e ‖	Global Effectivity Index
1	21	17.9253	6.8762	15.6341	0.8208
2	48	15.8911	5.0606	11.8287	0.8535
3	120	9.7408	2.2487	6.9199	0.8813
4	306	7.0493	0.2047	0.8497	0.9104
5	534	7.0422	0.1548	0.6392	0.9727
6	891	7.0424	0.1233	0.5085	0.9817
7	1491	7.0401	0.0988	0.4244	0.9999

TABLE -2

Numerical Test Results for Gradient of field Error Estimate

Mesh Number	No. of Elements	No. of Nodes	Global Effectivity Index
1	21	34	1.0867
2	30	47	1.1384
3	39	60	1.1676
4	126	167	0.9997
5	192	245	0.9998
6	309	378	0.9996
7	489	578	0.9998
8	759	872	0.9999

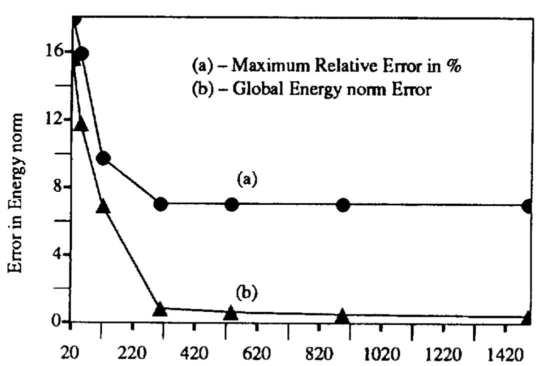

Fig. 1 : Error Convergence Plot – Post-Processing Error Estimate

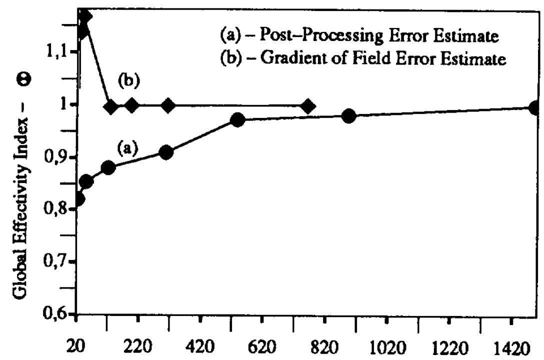

Fig. 2 : Asymptotic Convergence Plot of Effectivity Index – Θ

V. CONCLUSIONS

The reliability assessment of an *'a posteriori'* error estimate with a model problem is presented. The use of error measures to evaluate the reliability of an error estimate and the reliability measurement through asymptotic exactness of effectivity indices are analyzed. The reliability of two different error estimates in the adaptive computation of electromagnetic field problems is established through the asymptotic exactness of the computed effectivity indices. From the numerical test results we find that the gradient of field error estimate has a higher degree of reliability than the post-processing method of error estimation.

REFERENCES

[1] P. Fernandes, P. Girdinio, P. Molfino & M. Repetto, 'An Enchanced Error Estimator Procedure for Finite Element Field Computation with Adaptive Mesh Refinement', *IEEE Trans. on Magnetics*, Vol. 26, pp. 2187–2190 (1990).

[2] K. Adamiak, 'Local Error Indicator in Finite Element Analysis of Laplacian Fields on the Green Integration Formula', *Int. J. Numer. Methods. Eng.*, Vol. 33, pp. 1625–1642 (1992).

[3] K.C. Chellamuthu & Nathan Ida, 'A posteriori Error Indicator and Error Estimators for Adaptive Mesh Refinement', Unpublished.

[4] I. Babuska & W.C. Rheinboldt, 'Adaptive approaches and reliability Estimations in Finite element Analysis', *Comp. Methods in Applied Mech. and Eng.*, North Holland Pub. Company, pp. 519–540 (1979).

[5] I. Babuska & W.C. Rheinboldt, 'On the Reliability and Optimality of the Finite Element Method', *Computers & Structures*, Vol. 10, pp. 87–94 (1979).

[6] I. Babuska, L. Plank & R. Rodriquez, 'Quality Assessment of the a-posteriori Error Estimation in Finite Elements', *Finite Elements in Analysis and Design*, North-Holland, Vol. 11, pp. 285–306, (1992).

[7] P.G. Ciarlet, 'The Finite Element Method for Elliptic Problems', North-Holland Publishing Company, New York, (1978).

[8] I. Babuska & M. Suri, 'The Optimal Convergence Rate of the p-Version of the Finite Element Method', *SIAM J. Numer. Analysis*, Vol. 24, No. 4, pp. 750–776 (1987).

[9] A.K. Noor & I. Babuska, 'Quality Assessment and Control of Finite Element Solutions', *Finite Elements in Analysis and Design*, North-Holland, Vol. 3, pp. 1-26 (1987).

[10] R. Duran, M.A. Muschietti & R. Rodriguez, 'On the Asymptotic exactness of Error Estimators for Linear Triangular Finite Elements', *Numer. Math.*, Vol. 59, pp. 107-127 (1991).