Use of Reduced 3D Hexahedral Edge Elements for 2D TE Waveguides and Vector Potential Problems

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***Abstract* - We present in this paper the use of the 3D hexahedral edge element for two 2D applications. The procedure consists of first computing the 3D hexahedral edge elemental matrix. Then the "reduction" for 2D domains is applied in the assembling process, resulting in a simple way to compute 2D problems using edge elements without the need to define special 2D finite elements. The method maintains all the properties of a 2D method with the exception of the elemental matrix calculation.**

I. INTRODUCTION

Edge elements have been used successfully to solve some problems such as computation of waveguide modes where spurious modes are avoided. Also, in 3D low frequency problems using vector potential formulations, edge elements provide accurate results. However, the application of edge elements for 2D applications has the inherent difficulty of defining a simple and effective 2D element. In the procedure presented here these difficulties are eliminated by using a straightforward 3D hexahedral element. To apply this method, only the assembling is modified in order to match the necessary physical approach.

II. FIRST APPLICATION: TE WAVEGUIDE MODES

Computation of rectangular waveguide modes can be treated by the 2D approach presented here, in which the xy plane is the cross-section of the guide. For this type of waveguide, the propagating field is H_z while E_z is equal to zero. The required equation is

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + (\gamma^2 + k^2)H_z = 0 \qquad (1)$$

The analytical solution for the waveguide is

$$H_z(x,y) = H_0 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \qquad (2)$$

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where a and b are the dimensions of the waveguide. At cutoff, the propagation constant is zero $(\gamma=0)$ and the wavenumber $k_c^2=\omega^2\mu\varepsilon$ is used instead of the propagation constant. Substituting the solution in Eq. (2) into Eq. (1) at cutoff gives

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad (3)$$

The lowest mode of propagation can be obtained by:

$$k_c^2 = \left(\frac{\pi}{a}\right)^2 \quad for \quad m=1, \quad n=0 \quad if \quad a>b$$
$$k_c^2 = \left(\frac{\pi}{b}\right)^2 \quad for \quad m=0, \quad n=1 \quad if \quad b>a$$

The frequency is calculated from $k_c^2=(2\pi f)^2\mu\varepsilon$. Since Eq. (1) is similar to the well known *2D* vector potential formulation, nodal finite elements can be easily applied to solve it.

In applying edge elements[1-3] it is much more convenient to use the electric field **E** which only exists in the *xy* plane. The equivalent equation for this waveguide is

$$\nabla\times\nabla\times\mathbf{E} - k_c^2\mathbf{E} = 0 \qquad (4)$$

Application of Galerkin's method to this equation produces the matrix system

$$[A]\,[E] = k_c^2[B]\,[E] \qquad (5)$$

where [A] is generated from the first term in Eq. (4) and [B] from the the second term. To obtain the lowest frequency of propagation the eigenvalue problem in (5) is solved. To do so, Eq. (5) is written as

$$[A]^{-1}[A]\,[E] = k_c^2[A]^{-1}\,[B]\,[E]$$

Denoting $[C]=[A]^{-1}[B]$ and $k_c^2 = 1/\lambda$ gives

$$[C]\,[E] = \lambda[E] \qquad (6)$$

This is a standard eigenvalue problem. Application of the Power method yields the largest eigenvalue corresponding to the smallest k_c^2.[4]

III> APPLICATION OF 3D EDGE ELEMENTS

The waveguide is represented by a single layer of hexahedral elements as shown in Fig. 1, where there are $3x3$ elements, as an example. The height of the elements, "c" is chosen such that the element is close to a cube. The element is shown in Fig. 2 in local coordinates u,v,p, and the field **E** in the element is given as

$$E = \sum_{i=1}^{12} E_i \mathbf{w}_i$$

where E_i is the circulation of **E** on edge i, and $\mathbf{w}_i$ is the vector shape function defined in [5]. As an example, $\mathbf{w}_2=u(1-p)\nabla v$. Application of Galerkin's method to Eq. (4) yields the terms of matrices $[A]$ and $[B]$ in Eq. (5). These are required to obtain the matrix system in Eq. (6). The elemental contribution matrices have the following general terms:

$$a_{mn} = \int_{v_i} \frac{1}{\varepsilon_r}(\nabla\times\mathbf{w}_m)\cdot(\nabla\times\mathbf{w}_n)dv \qquad b_{mn} = \int_{v_i} k_0^2 \mu_r \mathbf{w}_m\cdot\mathbf{w}_n dv$$

where the indices m and n represent two edges of the element.

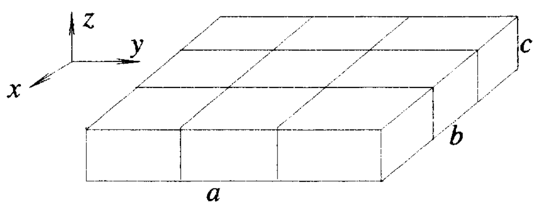

Fig. 1. A single layer mesh in the waveguide cross section.

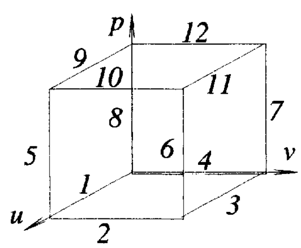

Fig. 2. The hexahedral element before reduction.

IV. REDUCTION TO 2D APPLICATIONS

The physical configuration treated here requires that the circulation of **E** on edges 5,6,7, and 8 (parallel to the z axis) be zero. Furthermore, the values of the unknowns on edges 1,2,3, and 4, should be the same as for edges 9,10, 11, and 12, respectively. A simple example to the system of equations is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad \begin{aligned} (a_{11} + a_{13})x_1 + a_{12}x_2 &= b_1 \\ (a_{21} + a_{23})x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

The system on the left can be simplified as shown if it is known apriori that $x_1=x_3$ and $b_1=b_3$. The application of this procedure for the problem can be performed easily in the elemental matrices $A(12,12)$ and $B(12,12)$ as follows:

- neglect all terms in columns and rows 5,6,7,8, because the values of the corresponding unknowns are zero.
- add the contributions into the following form:

$$\begin{bmatrix} a_{1,1}+a_{1,9} & a_{1,2}+a_{1,10} & a_{1,3}+a_{1,11} & a_{1,4}+a_{1,12} \\ a_{2,1}+a_{2,9} & a_{2,2}+a_{2,10} & \cdots & \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

The same transformation is applied to matrix B reducing $A(12,12)$ and $B(12,12)$ to matrices having dimensions $(4x4)$ since only edges 1,2,3, and 4 are used. A principal feature of this method is that only the unknowns of the lower plane $(p=0)$ are computed. Note also that since, for example, $a_{1,10}$ is equal to $a_{2,9}$, the symmetry of the matrix is maintained.

V. RESULTS

We apply the system for some rectangular waveguides obtaining the following table as a result

Waveguide size (m)	Theoretical value for k_0^2	$3x3$ mesh	$6x6$ mesh
$1x1$	9.87	10.8	10.09
$1x2$	2.487	2.7	2.52
$2x3$	1.096	1.2	1.12
$3x3$	1.096	1.2	1.12

The theoretical minimum value of k_0^2 is calculated as indicated in Eq. (3). As the mesh is refined, the lowest eigenvalue approaches the theoretical value.

VI. SECOND APPLICATION: 2D VECTOR POTENTIAL

In spite of the fact that the 2D vector potential formulations are well established, we will present an approach using edge elements. The classical vector potential problem is defined by the equation

$$\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial A}{\partial x}+\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial A}{\partial y}=-J \qquad (7)$$

where the current density J and A have only components in the z direction. Solving this problem means obtaining the magnitude of the z component of A for each node of the mesh while A_x and A_y are zero. This observation leads to solution of this problem with edge elements. In figure 1, if the height of the hexahedral element is equal to 1, we can obtain directly the required value of A_z, noting that with this kind of element the unknown is the circulation of the vector on the edge.

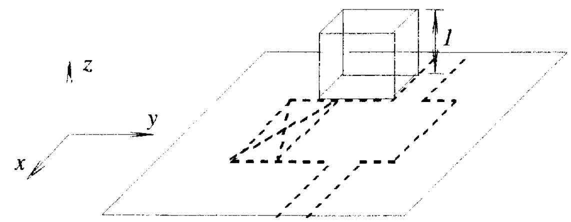

Fig. 3. Geometry with superimposed edge element.

Instead of solving Eq. (7) we will consider the complete 3D equation bellow:

$$\nabla\times\frac{1}{\mu}\nabla\times\mathbf{A}=\mathbf{J} \qquad (8)$$

The vector potential A is defined in the edge element of Fig. 2 as

$$\mathbf{A}=\sum_{i=1}^{12}\mathbf{w}_i A_i$$

and the unknowns to be considered are the values of A_i on the *12* edges. The application of the Galerkin method associated with the edge element yields the matrix system

$$[P]\,[A]=[S] \qquad (9)$$

where $[P]$ results from assembly of the left side terms of equation (8). The generic term of the elemental matrix (12×12) is:

$$p_{mn}=\int_v \frac{1}{\mu}\left(\nabla\times\mathbf{w}_m\right)\cdot\left(\nabla\times\mathbf{w}_n\right)dv \qquad (10)$$

The term of the source matrix $[S]$ is calculated as

$$s_m=\int_v \mathbf{J}\cdot\mathbf{w}_m dv \qquad (11)$$

VII. THE 2D APPLICATION

For the *2D* application it is clear that the circulation of A on the edges *1,2,3,4*, and *9,10,11,12* are zero, since A exists only on edges *5,6,7,8*, parallel to the z direction. We impose this condition by reducing the (12×12) elemental matrix to a (4×4) matrix, as well the source matrix, taking into account only the following contributions:

$$\begin{bmatrix} p_{55} & p_{56} & p_{57} & p_{58} \\ p_{65} & p_{66} & p_{67} & p_{68} \\ p_{75} & p_{76} & p_{77} & p_{78} \\ p_{85} & p_{86} & p_{87} & p_{88} \end{bmatrix} \qquad \begin{bmatrix} p_5 \\ p_6 \\ p_7 \\ p_8 \end{bmatrix} \qquad (12)$$

From the complete elemental matrices only the terms in Eq. (12) should be evaluated. In fact, in the 2D problem only the mesh in the xy plane is defined. If we associate each node with the corresponding edge emerging from this node, we preserve the same number of unknowns and bandwidth of the global matrices $[P]$ and $[S]$ as for the nodal element. In others words, a nodal element software can be easily employed, as well all the pre and post processors. The only change is in the evaluation of the elemental contribution matrix $[P]$ and the source vector $[S]$.

VIII. RESULTS

The problem in Fig. 4 was solved with nodal and edge elements. The graphical results for the two case shown in Fig. 4 are so close that visual inspection does not show any difference; the difference between maximum values of A is 1.9%. This example was performed to test the method in a realistic case, including airgaps, coils and iron; the number of nodes (or edges perpendicular to the xy plane) is *256*.

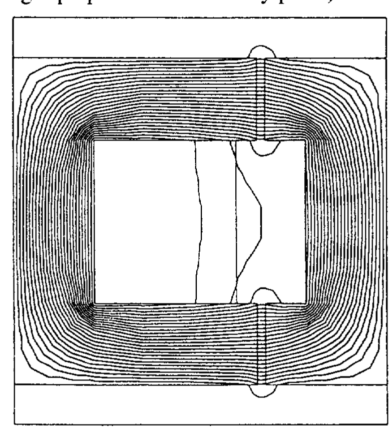

Fig. 4. Edge element solution using reduced 3-D elements.

To check accuracy, the problem in Fig. 4 was solved. In this case there is a conducting block with a current density J_z of *10 A/mm*^{2}. The analytic solution [6] gives the magnetic field energy as *2.2082 MJ/m.* Using a mesh with *13x13* nodes/edges we obtained *2.120* and *2.172 MJ/m* for nodal and edge formulations, yielding errors of *3.99%* and *1.64%*.

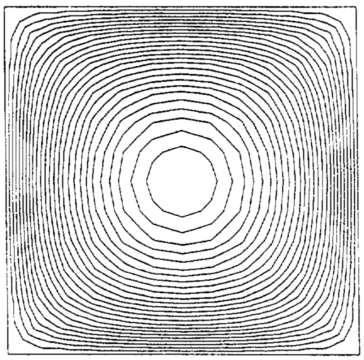

Fig. 4. Magnetic field in a current carrying block.

IX. CONCLUSIONS

An efficient and well known 3D edge element is applied to solve two different types of 2D applications: the problems of computing lowest eigenvalue for TE modes in rectangular waveguides and magnetostatic solutions using the magnetic vector potential. The calculation of elemental matrices is performed using the normal 3D hexahedral edge element. The reduction to a 2D application consists of modifying the assembly process. This method avoids the need to define special 2D edge elements and has all the advantages of 2D methods. The results presented here for the first application match the theoretical values for the given waveguide. For the second application, one of the examples shows noticeable improvement in accuracy. One of the advantages of this method is that existing software can be easily extended to include this technique. Pre- and post-processors can be used without modifications. Memory allocation is not affected either.

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