Time domain surface impedance concept for low frequency electromagnetic problems—Part I: Derivation of high order surface impedance boundary conditions in the time domain

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Abstract: The surface impedance boundary conditions (SIBCs) for the tangential component of the electric field and normal component of the magnetic field on the smooth curved surface of a homogeneous non-magnetic conductor are derived in time- and frequency-domain. Scale factors for the basic variables are introduced in such a way that, a small parameter, equal to the ratio of the penetration depth and the body's characteristic size, appears in the dimensionless Maxwell's equations for the conducting region. The perturbation method is then used to represent the SIBCs in the form of power series in this small parameter and the first four terms of the expansions are derived. The zero-order, first-order, second-order and third-order terms of the expansions are the solution of the problem in the perfect electrical conductor limit, the Leontovich approximation, the Mitzner approximation and in the high order approximation (referred to as Rytov's approximation), respectively. Therefore, the accuracy of the proposed conditions exceeds the accuracy of the SIBC for planar surfaces (Leontovich's approximation) that are usually used in the time-domain analysis, by two orders of magnitude. In Part II of this paper, the formulation of the SIBCs developed here in conjunction with a boundary element method is demonstrated and applied to the problem of transient skin and proximity effect problems in cylindrical conductors.

1 Introduction

In most electromagnetic problems the space under consideration consists of several media. The electromagnetic field governing equations written for each region are linked by the boundary conditions involving values on both sides of the interface. Thus one has to solve the problem for all media simultaneously even if the main focus of interest is on only one of them. However, in some particular cases the number of regions involved in the solution procedure may be reduced. A classical example is elimination of a body of infinite conductivity or perfect electrical conductor (PEC) from the computational space by enforcing the tangential electric field or normal magnetic flux to be equal to zero at the boundary (so-called PEC boundary condition):

$$\vec{n} \times \vec{E}\Big|_{interface} = 0; \quad \vec{n} \cdot \vec{B}\Big|_{interface} = 0 \tag{1}$$

In practice, any real material has finite conductivity so that the perfect electrical conductor is no more than a model of a good conductor in which the skin depth is assumed to be zero. Although the PEC condition is very attractive for implementation, the diffusion of the electromagnetic field into conductors may be neglected only in a limited number of cases. For this reason, the application of the PEC limit is of limited scope. For example, the electromagnetic penetration depth δ in copper at an incident frequency of 1 MHz is about 2×10^{-4} m. Is this skin layer thin or thick? Obviously, the question is meaningful only if another quantity is specified so that the two can be compared. One quantity that may be used for this purpose is the characteristic size D of the conductor. In our example, the penetration depth in the conductor is definitely not small if the conductor's thickness equals 5×10^{-5} m (a typical thickness in printed circuit board technology) and the PEC condition may not be applied in this case. One may expect that the use of the PEC condition will lead to errors proportional to δ/D.

Since the PEC is a limiting case of a real conductor, it is natural to expect that the PEC condition is also a particular case of a more general approximate boundary condition relating electromagnetic quantities at the conductor/dielectric interface. Existence of this approximate boundary condition follows directly from Snell's law of refraction: if the electromagnetic wave propagates from a low conductivity medium to a high conductivity medium, the reflection angle is about 90 degrees and is practically independent of the angle of incidence. Suppose the conducting region is so large that the wave attenuates completely inside the region. Then the electromagnetic field distribution in the conductor's skin layer can be described as a damped plane wave propagating into the depth of the conductor, normal to its surface. In other words, the behavior of the electromagnetic field in the conducting region may be assumed to be known *a priori* as in the case of the PEC. The electromagnetic field is continuous across the real conductor's surface so the

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intrinsic impedance of the wave remains the same at the interface. Therefore, the ratio E_x/H_y at the xy-plane of the dielectric/conductor interface is assumed to be equal to the intrinsic impedance of the plane wave propagating in the conductor in the positive z-direction:

$$\left.\frac{E_x}{H_y}\right|_{interface} = Z_\omega = \left.\sqrt{\frac{j\omega_{source}\mu_{cond}}{\sigma_{cond} + j\omega_{source}\epsilon_{cond}}}\right|_{\sigma\gg\omega\epsilon} \approx \frac{1+j}{2}\omega_{source}\mu_{cond}\delta; \quad \delta = \sqrt{\frac{2}{\omega_{source}\sigma_{cond}\mu_{cond}}} \tag{2}$$

The relationship in (2) does not depend on coordinates and therefore the surface impedance is assumed to be constant over the conductor's surface.

The surface relationship in (2), taking into account the parameters of the conductor's material and the source, contains all necessary information about the field distribution in the conductor's volume. Thus it may be used as a boundary condition to the governing equations for the dielectric space thereby excluding the conductor from the region of solution. This is the basic idea of the surface impedance concept. Historically (2) is associated with the name of Leontovich although he proposed a more accurate condition with first order correction terms that accounted for the curvature of the interface [1]. The actual term 'surface impedance' was introduced by Schelkunoff [2] by analogy with the ratio of voltage and current in circuit theory. The surface impedance vanishes as the conductivity tends to infinity and the skin depth tends to zero. In this limiting case (2) reduces to the PEC condition. The presence of δ in (2) leads us to expect that the approximation error of Leontovich's condition should be of the order of magnitude δ^2/D^2.

From the point of view of mathematical physics, the surface impedance concept means that in the skin layer the variation of the field along the surface is assumed small compared to the variation in the normal direction. Thus the field derivatives in the directions tangential to the surface may be neglected compared to the normal derivative and the original 3-dimensional equation of the field diffusion into the conductor is reduced to a 1-dimensional problem [3]. This is frequently referred to as the skin effect approximation [4]. Note that the same approximation is at the root of the theory of boundary layers in fluid mechanics. In contrast to (1), which is a Dirichlet boundary condition, (2) can be considered as an additional equation relating different unknowns at the interface.

Although the idea is rather transparent, several generations separate Snell's laws and the surface impedance concept, which has been formulated only in the late 1930 s when the then newly emerging radio technology required the development of a theory of propagation of electromagnetic waves of an antenna over the earth's surface [5, 6]. An analytic solution for the particular case of a vertical dipole radiating over the conducting half-space was obtained by Sommerfeld [7], but the general problem involving layered media separated by curved interfaces has not been solved so far. Leontovich proposed a different approach, namely to restrict the general problem by considering the practically important air region using the surface impedance boundary condition (SIBC) at the air/earth interface. Of course, the earth is not a good conductor in the common sense. However, the characteristic dimensions in this class of electromagnetic problems are large enough for attenuation of the wave in the earth so that the surface impedance technique may be applied.

The first rigorous mathematical analysis has been done by Rytov who sought the solution in the form of power series in the skin depth δ [8]: $\vec{E} = \sum_{i=0}^{\infty} \delta^i \vec{e}_i(\vec{r}) \exp[\phi(\eta/\delta)]$; $\vec{H} = \sum_{i=0}^{\infty} \delta^i \vec{h}_i(\vec{r}) \exp[\phi(\eta/\delta)]$ where η is the co-ordinate directed into the conductor normal to its surface, $\vec{e}_i$ and $\vec{h}_i$ are unknown coefficients and ϕ is an unknown function. Substituting the expansions in Maxwell's equations and equating the coefficients of equal powers of δ, the solutions for $\vec{e}_i$, $\vec{h}_i$ and ϕ are obtained. It is easy to see that the Rytov expansions generalise the solution of the 1-dimensional problem of the magnetic field diffusion into the conductor normal to its surface. It is important to emphasise that this solution must be known *a priori*. The first order terms of the expansions (actually, first non-zero terms) gave Leontovich's condition. Thus improvements in Leontovich's condition can be obtained by derivation of the next higher order terms of the expansions. Rytov also stated the problems of calculation of the surface impedance at curved interfaces and non-homogeneous conductors. Unfortunately, Rytov's contribution is not as well known as Leontovich's work.

A further improvement of practical importance has been introduced by Mitzner, [9] who developed an SIBC, known by his name, to any smooth conducting surface by introducting terms of the order $(\delta/D)^2$, allowing for the conductor's curvature. Although Mitzner derived the SIBC in his own way, calculation of the second order terms in Rytov's expansions leads to the same result. A fundamentally different case is the vicinity of a conducting edge where the magnetic field distribution is singular and the diffusion may not be described by a one-dimensional equation. Numerous semi-empirical attempts have been made to modify Leontovich's condition near corners [10–13], but a rigorous asymptotic solution was obtained using the perturbation approach [14] where the field distribution around a perfectly conducting edge is used as a boundary condition to the 2-D or 3-D problem for the field inside the real conductor to derive the first-order SIBC near the edge. Another family of surface impedance boundary conditions has been developed for layered structures [15–18] but its application area is limited to high frequency problems and therefore is outside the scope of this paper.

The surface impedance concept can also be used in transient problems, when, for instance, the current pulse duration is so short that the field has no time to diffuse deep into the conductor and remains concentrated near the surface [19]. There are two basic approaches to solve transient problems: (a) by obtaining the solution in the frequency domain for the time-harmonic exciting source and using inverse Fourier transform techniques to calculate the required transient data and (b) by formulating the problem directly in the time domain. The second method is gaining acceptance with the advent of fast computers. Moreover, in some cases the time domain approach is more natural from a physical point of view as it takes place in nonlinear problems. In the general case neither magnetic nor electric fields in the conductor with nonlinear properties of materials are actually harmonic so that use of models based on a single frequency is not feasible and recently time domain SIBCs for nonlinear problems have been developed [20].

The time domain form of Leontovich's condition is obtained from (2) using the Laplace transform:

$$E_x|_{interface} = Z_t^* H_y|_{interface}; \quad Z_t|_{\sigma\gg\omega\epsilon} = \sqrt{\mu/(2\pi\sigma t^3)} \tag{3}$$

where the sign '$*$' denotes a time domain convolution product.

Surprisingly, the low order SIBC (3) is widely used to the present day despite the fact that higher approximation order allows extension of the range of problems for which the surface impedance concept can be applied.

The purpose of this paper is a rigorous derivation and analysis of the surface impedance boundary conditions of high order of approximation for any time-dependence of the electromagnetic source in the low frequency case. In this work we generalise the Rytov approach by using the small parameter based on the duration of the incident pulse of an arbitrary shape. In this case not only the distribution of the field inside the conductor normal to its surface may not be known *a priori* unlike the time harmonic case considered in the Rytov work, but even the 1-D equation of diffusion must be derived as part of the solution process. It extends the application area of the proposed approach allowing rigorous derivation of the SIBCs for non linear and non-homogeneous conductors. The following basic steps are developed below:

1. We introduce the dimensionless variables related to the boundary layer near the body's surface in such a way, that the small parameter p, equal to the ratio of the transient electromagnetic penetration depth and the characteristic size of the body, appears explicitly in the governing differential field equations for the conducting region and in the surface integral equations for free space (Sections 3 and 4).
2. We represent the electric and magnetic fields inside the body as a power series in the small parameter p. The expansions are substituted into the equations for the initial functions and the 1-D reduced problems for the terms of expansions are obtained (Section 5).
3. From the formulations for the terms of the order p, p^2 and p^3 we obtain the tangential component of the magnetic field and the normal component of the electric field on the surface of the body in the Laplace domain (Section 6).
4. By using the inverse Laplace transform, the time domain SIBC of the order p, p^2 and p^3 are obtained (Section 7).

2 Statement of the problem and basic equations

Consider a homogeneous body of finite conductivity $(\epsilon_1, \mu_1, \sigma_1)$, surrounded by a non-conductive medium $(\epsilon_2, \mu_2, \sigma_2 = 0)$. The parameters of the conducting material are assumed to be constant. Let the characteristic dimension of the problem be small compared with the wavelength of the incident electromagnetic field. Thus the electric field $\vec{E}$ and magnetic field $\vec{H}$ inside the body can be described by Maxwell's equations neglecting the displacement current density:

$$\nabla \times \vec{E} = -\mu_1 \frac{\partial \vec{H}}{\partial t} \tag{4}$$

$$\nabla \times \vec{H} = \sigma \vec{E} \tag{5}$$

$$\nabla \cdot \vec{H} = 0 \tag{6}$$

The exact boundary conditions at the surface of the conductor are

$$\begin{gathered} \vec{n} \times \vec{H}\big|_{cond} = \vec{n} \times \vec{H}\big|_{diel}; \quad \mu_1 \vec{n} \cdot \vec{H}\big|_{cond} = \mu_2 \vec{n} \cdot \vec{H}\big|_{diel}; \\ \vec{n} \times \vec{E}\big|_{cond} = \vec{n} \times \vec{E}\big|_{diel}; \quad \vec{n} \cdot \vec{E}\big|_{cond} = \vec{n} \cdot \vec{E}\big|_{diel} \end{gathered} \tag{7}$$

Let the time variation of the incident field be such that the electromagnetic penetration depth δ into the body remains small compared with the characteristic dimension D of the surface of the body

$$\delta = \sqrt{\tau/\sigma\mu_1} \ll D \tag{8}$$

where τ is the ratio $2/\omega$ in the case of time-harmonic fields or the incident pulse duration in the case of transient sources. The presence of the condition (8) makes it possible to transform (4)–(6) by using asymptotic expansion techniques with the purpose of deriving the normal components of the magnetic field and the tangential components of the electric field at the surface of the body in explicit form.

3 Local co-ordinates

Following Mitzner's approach of deriving the SIBC with allowance for the curvature of the body's surface, we re-write (4)–(6) in the local quasi-spherical orthogonal curvilinear system $(\alpha_1, \alpha_2, \eta)$, related to the body surface (Fig. 1):

$$\begin{gathered} \frac{\partial(e_{\alpha_k} H_{\alpha_k})}{\partial \eta} - \frac{\partial(e_\eta H_\eta)}{\partial \alpha_k} = (-1)^{3-k} e_{\alpha_k} e_\eta \sigma E_{\xi_{3-k}}, \ k = 1, 2; \\ \sum_{i=1}^{2} (-1)^i \frac{\partial(e_{\alpha_i} H_{\alpha i})}{\partial \alpha_{\alpha_{3-i}}} = e_{\alpha_1} e_{\alpha_2} \sigma E_\eta \end{gathered} \tag{9}$$

$$\begin{gathered} \frac{\partial(e_{\alpha_k} E_{\alpha_k})}{\partial \eta} - \frac{\partial(e_\eta E_\eta)}{\partial \alpha_k} = (-1)^k e_{\alpha_k} e_\eta \mu_1 \frac{\partial H_{\xi_{3-k}}}{\partial t}, \ k = 1, 2; \\ \sum_{i=1}^{2} (-1)^{3-i} \frac{\partial(e_{\alpha_i} E_{\alpha_i})}{\partial \alpha_{\alpha_{3-i}}} = e_{\alpha_1} e_{\alpha_2} \mu_1 \frac{\partial H_\eta}{\partial t} \end{gathered} \tag{10}$$

$$\sum_{i=1}^{2} \frac{\partial(e_{\alpha_i} e_\eta H_{\alpha_i})}{\partial \alpha_{\alpha_{3-i}}} + \frac{\partial(e_{\alpha_1} e_{\alpha_2} H_\eta)}{\partial \eta} = 0 \tag{11}$$

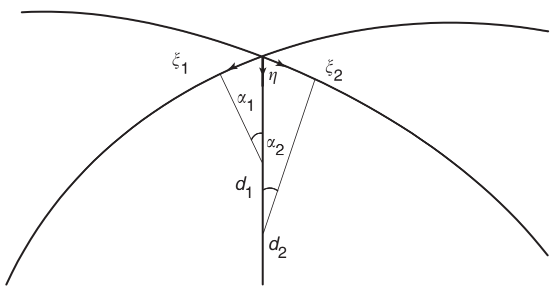

Fig. 1 *Local orthogonal curvilinear co-ordinate systems related to the surface*

where $e_{\alpha_1}, e_{\alpha_2}, e_\eta$ are the Lame coefficients. The co-ordinates α_1 and α_2 are defined as angles whereas the co-ordinate η is defined as the distance measured from the surface to a current point inside the body. The Lame coefficients of the co-ordinates $(\alpha_1, \alpha_2, \eta)$ are written in the form:

$$e_{\alpha_1} = d_1 - \eta; \quad e_{\alpha_2} = d_2 - \eta; \quad e_\eta = 1 \tag{12}$$

where $d_k, k = 1, 2$, are the local radii of curvature of the corresponding co-ordinate line.

In analysis of the skin effect problem using the perturbation technique, it is natural to perform derivations in the system where all co-ordinates are of the same dimensionality. For this purpose we replace the co-ordinates α_1 and α_2 (angles) by ξ_1 and ξ_2 (distances) (Fig. 1). The characteristic lengths associated with the co-ordinates ξ_i and η are D and

δ, respectively. The two co-ordinate systems are related as follows:

$$\xi_1 = d_1\alpha_1; \quad \xi_2 = d_2\alpha_2 \tag{13}$$

Equations (9)–(11) with the co-ordinates in (13) are written in the form

$$\frac{\partial H_{\xi_k}}{\partial \eta} - \frac{d_k}{d_k-\eta}\frac{\partial H_\eta}{\partial \xi_k} - \frac{H_{\xi_k}}{d_k-\eta} = (-1)^{3-k}\sigma E_{\xi_{3-k}}, \quad k=1,2;$$
$$\sum_{i=1}^{2}(-1)^i \frac{d_{3-i}}{d_{3-i}-\eta}\frac{\partial H_{\xi_i}}{\partial \xi_{3-i}} = \sigma E_\eta \tag{14}$$

$$\frac{\partial E_{\xi_k}}{\partial \eta} - \frac{d_k}{d_k-\eta}\frac{\partial E_\eta}{\partial \xi_k} - \frac{E_{\xi_k}}{d_k-\eta} = (-1)^{k}\mu_1\frac{\partial H_{\xi_{3-k}}}{\partial t}, \quad k=1,2;$$
$$\sum_{i=1}^{2}(-1)^{3-i} \frac{d_{3-i}}{d_{3-i}-\eta}\frac{\partial E_{\xi_i}}{\partial \xi_{3-i}} = \mu_1\frac{\partial H_\eta}{\partial t} \tag{15}$$

$$\frac{\partial H_\eta}{\partial \eta} + \sum_{i=1}^{2}\frac{d_i}{d_i-\eta}\frac{\partial H_{\xi_i}}{\partial \xi_i} = H_\eta\sum_{i=1}^{2}(d_i-\eta)^{-1} \tag{16}$$

4 Dimensionless variables

Following perturbation theory methods, we introduce the characteristic scale factors for the variables of the problem. The choice of the scale factors for co-ordinates ξ_1, ξ_2, η is evident, namely the characteristic size $D = \min(d_1, d_2)$ of the conductor's surface for ξ_1 and ξ_2, and the penetration depth δ for η. The quantity τ was introduced in (8) as the ratio $2/\omega$ in the case of time-harmonic fields or the incident pulse duration in the case of transient sources and is therefore the natural scale factor for time. We denote for now the scale factors for the electric and magnetic fields as E^* and H^*, respectively.

Let us now switch to the following non-dimensional variables:

$$\tilde{\xi}_k = \xi_k/D;\ \tilde{t} = t/\tau;\ \tilde{\eta} = \eta/\delta;\ \tilde{E} = E/E^*;\ \tilde{H} = H/H^* \tag{17}$$

Here and below, the sign '$\sim$' denotes non-dimensional quantities.

Substitution of (17) into (14)–(16) gives

$$\frac{H^*}{\delta}\left(\frac{\partial \tilde{H}_{\xi_k}}{\partial \tilde{\eta}} - \frac{\delta}{D}\frac{d_k}{d_k-\delta\tilde{\eta}}\frac{\partial \tilde{H}_\eta}{\partial \tilde{\xi}_k} - \frac{\tilde{H}_{\xi_k}}{d_k-\delta\tilde{\eta}}\right) = (-1)^{3-k}E^*\sigma\tilde{E}_{\xi_{3-k}}, \quad k=1,2 \tag{18a}$$

$$\frac{H^*}{D}\sum_{i=1}^{2}(-1)^i\frac{d_{3-i}}{d_{3-i}-\delta\tilde{\eta}}\frac{\partial \tilde{H}_{\xi_i}}{\partial \tilde{\xi}_{3-i}} = \sigma E^*\tilde{E}_\eta \tag{18b}$$

$$\frac{E^*}{D}\left(\frac{D}{\delta}\frac{\partial \tilde{E}_{\xi_k}}{\partial \tilde{\eta}} - \frac{d_k}{d_k-\delta\tilde{\eta}}\frac{\partial \tilde{E}_\eta}{\partial \tilde{\xi}_k} - \frac{\tilde{E}_{\xi_k}}{d_k-\delta\tilde{\eta}}\right) = (-1)^k\mu_1\frac{H^*}{\tau}\frac{\partial \tilde{H}_{\xi_{3-k}}}{\partial \tilde{t}}, \quad k=1,2 \tag{19a}$$

$$\frac{E^*}{D}\sum_{i=1}^{2}(-1)^{3-i}\frac{d_{3-i}}{d_{3-i}-\delta\tilde{\eta}}\frac{\partial \tilde{E}_{\xi_i}}{\partial \tilde{\xi}_{3-i}} = \mu_1\frac{H^*}{\tau}\frac{\partial \tilde{H}_\eta}{\partial \tilde{t}} \tag{19b}$$

$$\frac{D}{\delta}\frac{\partial \tilde{H}_\eta}{\partial \tilde{\eta}} + \sum_{i=1}^{2}\frac{d_i}{d_i-\delta\tilde{\eta}}\frac{\partial \tilde{H}_{\xi_i}}{\partial \tilde{\xi}_i} = D\tilde{H}_\eta\sum_{i=1}^{2}(d_i-\delta\tilde{\eta})^{-1} \tag{20}$$

Note that not all scale factors in (18)–(20) are actually independent. Indeed, from (8) it follows that

$$\delta = \sqrt{\tau/(\sigma\mu_1)} = \sqrt{\tau/(\sigma\mu_1 D^2)}D = pD;$$
$$p = \delta/D = \sqrt{\tau/(\sigma\mu_1 D^2)} \ll 1 \tag{21}$$

where the parameter p is proportional to the ratio of the penetration depth and characteristic size of the conductor's surface. The relation between E^* and H^* follows from (19):

$$E^* = \frac{\mu_1 D}{\tau}H^* \tag{22}$$

Therefore, the total number of basic scale factors is 3, for example: τ, D and H^*. The practical selection of the scale factors should be based on the input data of a given problem [21]. Sometimes the total current is used as input data. In such cases it is natural to use the characteristic current I^* as one of the basic scale factors instead of E^* or H^*. The relation between I^* and H^* can be obtained using the Biot-Savart law

$$\vec{H} = (4\pi)^{-1}\int_L \vec{I}\times(\vec{R}/R^3)dl \tag{23}$$

Transfer to the non-dimensional variables in (23) gives:

$$\tilde{\vec{H}}H^* = I^*(4\pi D)^{-1}\int_L \tilde{\vec{I}}\times\left(\tilde{\vec{R}}/\tilde{R}^3\right)d\tilde{l} \tag{24}$$

Thus

$$H^* = I^*/(4\pi D^*) \tag{25}$$

Substitution of (21)–(22) into (18)–(19) gives

$$p\frac{\partial \tilde{H}_{\xi_k}}{\partial \tilde{\eta}} - \frac{p^2\tilde{H}_{\xi_k}}{\tilde{d}_k-p\tilde{\eta}} - \frac{p^2\tilde{d}_k}{\tilde{d}_k-p\tilde{\eta}}\frac{\partial \tilde{H}_\eta}{\partial \tilde{\xi}_k} = (-1)^{3-k}\tilde{E}_{\xi_{3-k}},\ k=1,2;$$
$$p^2\sum_{i=1}^{2}(-1)^i\frac{\tilde{d}_{3-i}}{\tilde{d}_{3-i}-\tilde{\eta}}\frac{\partial \tilde{H}_{\xi_i}}{\partial \tilde{\xi}_{3-i}} = \tilde{E}_\eta \tag{26}$$

$$\frac{\partial \tilde{E}_{\xi_k}}{\partial \tilde{\eta}} - \frac{p\tilde{E}_{\xi_k}}{\tilde{d}_k-p\tilde{\eta}} - \frac{p\tilde{d}_k}{\tilde{d}_k-p\tilde{\eta}}\frac{\partial \tilde{E}_\eta}{\partial \tilde{\xi}_k} = (-1)^k p\frac{\partial \tilde{H}_{\xi_{3-k}}}{\partial \tilde{t}},\ k=1,2;$$
$$p^2\sum_{i=1}^{2}(-1)^{3-i}\frac{\tilde{d}_{3-i}}{\tilde{d}_{3-i}-\tilde{\eta}}\frac{\partial \tilde{E}_{\xi_i}}{\partial \tilde{\xi}_{3-i}} = \frac{\partial \tilde{H}_\eta}{\partial t} \tag{27}$$

$$\frac{\partial \tilde{H}_\eta}{\partial \tilde{\eta}} - p\tilde{H}_\eta\sum_{i=1}^{2}\frac{1}{\tilde{d}_i-p\tilde{\eta}} = -p\sum_{i=1}^{2}\frac{\tilde{d}_i}{\tilde{d}_i-p\tilde{\eta}}\frac{\partial \tilde{H}_{\xi_i}}{\partial \tilde{\xi}_i} \tag{28}$$

where $\tilde{d}_k = d_k/D$, $k=1,2$. Equations (26)–(28) do not contain the scale factors for the electric and magnetic fields anymore. The remaining scale factors for the co-ordinates are included only in the form of the ratio δ/D which is the small parameter of the problem.

5 Expansions in the small parameter

We now apply Laplace's transform following the rule $\overline{\tilde{f}}(\tilde{s}) = \int_0^\infty \tilde{f}(\tilde{t})\exp(-\tilde{s}\tilde{t})d\tilde{t}$. Here f denotes an arbitrary function. In the Laplace-domain, (26)–(28) are written in the form:

$$p\frac{\partial\overline{\overline{H}}_{\xi_k}}{\partial\tilde{\eta}}-\frac{p^2\overline{\overline{H}}_{\xi_k}}{\tilde{d}_k-p\tilde{\eta}}-\frac{p^2\tilde{d}_k}{\tilde{d}_k-p\tilde{\eta}}\frac{\partial\overline{\overline{H}}_{\eta}}{\partial\tilde{\xi}_k}=(-1)^{3-k}\overline{\overline{E}}_{\xi_{3-k}},\quad k=1,2;$$
$$p^2\sum_{i=1}^{2}(-1)^i\frac{\tilde{d}_{3-i}}{\tilde{d}_{3-i}-\tilde{\eta}}\frac{\partial\overline{\overline{H}}_{\xi_i}}{\partial\tilde{\xi}_{3-i}}=\overline{\overline{E}}_{\eta} \tag{29}$$

$$\frac{\partial\overline{\overline{E}}_{\xi_k}}{\partial\tilde{\eta}}-\frac{p\overline{\overline{E}}_{\xi_k}}{\tilde{d}_k-p\tilde{\eta}}-\frac{p\tilde{d}_k}{\tilde{d}_k-p\tilde{\eta}}\frac{\partial\overline{\overline{E}}_{\eta}}{\partial\tilde{\xi}_k}=(-1)^k p\tilde{s}\overline{\overline{H}}_{\xi_{3-k}},\ k=1,\ 2;$$
$$p^2\sum_{i=1}^{2}(-1)^{3-i}\frac{\tilde{d}_{3-i}}{\tilde{d}_{3-i}-\tilde{\eta}}\frac{\partial\overline{\overline{E}}_{\xi_i}}{\partial\tilde{\xi}_{3-i}}=\tilde{s}\overline{\overline{H}}_{\eta} \tag{30}$$

$$\frac{\partial\overline{\overline{H}}_{\eta}}{\partial\tilde{\eta}}-p\overline{\overline{H}}_{\eta}\sum_{i=1}^{2}\frac{1}{\tilde{d}_i-p\tilde{\eta}}=-p\sum_{i=1}^{2}\frac{\tilde{d}_i}{\tilde{d}_i-p\tilde{\eta}}\frac{\partial\overline{\overline{H}}_{\xi_i}}{\partial\tilde{\xi}_i} \tag{31}$$

Since the parameter p is small, we represent the functions, for which the solutions are sought, in the form of asymptotic expansions in the parameter p:

$$\vec{\tilde{\overline{H}}}=\sum_{m=0}^{\infty}p^m\vec{\tilde{\overline{H}}}_m;\quad \vec{\tilde{\overline{E}}}=\sum_{m=0}^{\infty}p^m\vec{\tilde{\overline{E}}}_m \tag{32}$$

The following functions can also be represented as expansions in the small parameter p:

$$\frac{1}{\tilde{d}_k-p\tilde{\eta}}=\frac{1}{\tilde{d}_k}+p\frac{\tilde{\eta}}{\tilde{d}_k^2}+p^2\frac{\tilde{\eta}^2}{\tilde{d}_k^3}+O(p^3)$$
$$\frac{\tilde{d}_k}{\tilde{d}_k-p\tilde{\eta}}=1+p\frac{\tilde{\eta}}{\tilde{d}_k}+p^2\frac{\tilde{\eta}^2}{\tilde{d}_k^2}+O(p^3) \tag{33}$$

Substituting the expansions (32) and (33) into (29)–(31) and equating the coefficients of equal powers of p, the following equations for the expansion coefficients are obtained:

$m=0$:

$$(\vec{\tilde{\overline{E}}}_0)_{\xi_1}=(\vec{\tilde{\overline{E}}}_0)_{\xi_2}=(\vec{\tilde{\overline{E}}}_0)_{\eta}=(\vec{\tilde{\overline{H}}}_0)_{\eta}=0 \tag{34}$$

$m=1$:

$$\frac{\partial(\vec{\tilde{\overline{E}}}_1)_{\xi_k}}{\partial\tilde{\eta}}=(-1)^k\tilde{s}(\vec{\tilde{\overline{H}}}_0)_{\xi_{3-k}}\quad k=1,2 \tag{35a}$$

$$(\vec{\tilde{\overline{E}}}_1)_{\xi_k}=(-1)^k\frac{\partial(\vec{\tilde{\overline{H}}}_0)_{\xi_{3-k}}}{\partial\tilde{\eta}}\quad k=1,2 \tag{35b}$$

$$\sum_{i=1}^{2}(-1)^i\frac{\partial(\vec{\tilde{\overline{E}}}_1)_{\xi_{3-k}}}{\partial\tilde{\xi}_i}=\tilde{s}(\vec{\tilde{\overline{H}}}_1)_{\eta} \tag{35c}$$

$$\frac{\partial(\vec{\tilde{\overline{H}}}_1)_{\eta}}{\partial\eta}=-\sum_{i=1}^{2}\frac{\partial(\vec{\tilde{\overline{H}}}_0)_{\xi_i}}{\partial\xi_i} \tag{35d}$$

$$(\vec{\tilde{\overline{E}}}_1)_{\eta}=0 \tag{35e}$$

$m=2$:

$$\frac{\partial(\vec{\tilde{\overline{E}}}_2)_{\xi_k}}{\partial\tilde{\eta}}=\frac{(\vec{\tilde{\overline{E}}}_1)_{\xi_k}}{\tilde{d}_k}+(-1)^k\tilde{s}(\vec{\tilde{\overline{H}}}_1)_{\xi_{3-k}}\quad k=1,2 \tag{36a}$$

$$(\vec{\tilde{\overline{E}}}_2)_{\xi_k}=(-1)^k\left[\frac{\partial(\vec{\tilde{\overline{H}}}_1)_{\xi_{3-k}}}{\partial\tilde{\eta}}-\frac{(\vec{\tilde{\overline{H}}}_0)_{\xi_{3-k}}}{\tilde{d}_{3-k}}\right]\quad k=1,2 \tag{36b}$$

$$\sum_{i=1}^{2}(-1)^i\left[\frac{\partial(\vec{\tilde{\overline{E}}}_2)_{\xi_{3-i}}}{\partial\tilde{\xi}_i}+\frac{\tilde{\eta}}{\tilde{d}_i}\frac{\partial(\vec{\tilde{\overline{E}}}_1)_{\xi_{3-i}}}{\partial\tilde{\xi}_i}\right]=\tilde{s}(\vec{\tilde{\overline{H}}}_2)_{\eta} \tag{36c}$$

$$(\vec{\tilde{\overline{E}}}_2)_{\eta}=\sum_{i=1}^{2}(-1)^{3-i}\frac{\partial(\vec{\tilde{\overline{H}}}_0)_{\xi_{3-i}}}{\partial\xi_i} \tag{36d}$$

$$\frac{\partial(\vec{\tilde{\overline{H}}}_2)_{\eta}}{\partial\tilde{\eta}}=(\vec{\tilde{\overline{H}}}_1)_{\eta}\sum_{i=1}^{2}\tilde{d}_i^{-1}-\sum_{i=1}^{2}\left[\frac{\partial(\vec{\tilde{\overline{H}}}_1)_{\xi_i}}{\partial\tilde{\xi}_i}+\frac{\tilde{\eta}}{\tilde{d}_i}\frac{\partial(\vec{\tilde{\overline{H}}}_0)_{\xi_i}}{\partial\tilde{\xi}_i}\right] \tag{36e}$$

$m=3$:

$$\frac{\partial(\vec{\tilde{\overline{E}}}_3)_{\xi_k}}{\partial\tilde{\eta}}=\frac{(\vec{\tilde{\overline{E}}}_2)_{\xi_k}}{\tilde{d}_k}+\tilde{\eta}\frac{(\vec{\tilde{\overline{E}}}_1)_{\xi_k}}{\tilde{d}_k^2}+\frac{\partial(\vec{\tilde{\overline{E}}}_2)_{\eta}}{\partial\tilde{\xi}_k}+(-1)^k\tilde{s}(\vec{\tilde{\overline{H}}}_2)_{\xi_{3-k}}$$
$$k=1,2 \tag{37a}$$

$$(\vec{\tilde{\overline{E}}}_3)_{\xi_k}=(-1)^k\left[\frac{\partial(\vec{\tilde{\overline{H}}}_2)_{\xi_{3-k}}}{\partial\tilde{\eta}}-\frac{(\vec{\tilde{\overline{H}}}_1)_{\xi_{3-k}}}{\tilde{d}_{3-k}}-\tilde{\eta}\frac{(\vec{\tilde{\overline{H}}}_0)_{\xi_{3-k}}}{\tilde{d}_{3-k}^2}-\frac{\partial(\vec{\tilde{\overline{H}}}_1)_{\eta}}{\partial\tilde{\xi}_{3-k}}\right]$$
$$k=1,2 \tag{37b}$$

$$\sum_{i=1}^{2}(-1)^i\left[\frac{\partial(\vec{\tilde{\overline{E}}}_3)_{\xi_{3-i}}}{\partial\tilde{\xi}_i}+\frac{\tilde{\eta}}{\tilde{d}_i}\frac{\partial(\vec{\tilde{\overline{E}}}_2)_{\xi_{3-i}}}{\partial\tilde{\xi}_i}+\frac{\tilde{\eta}^2}{\tilde{d}_i^2}\frac{\partial(\vec{\tilde{\overline{E}}}_1)_{\xi_{3-i}}}{\partial\tilde{\xi}_i}\right]$$
$$=\tilde{s}(\vec{\tilde{\overline{H}}}_3)_{\eta} \tag{37c}$$

$$(\vec{\tilde{\overline{E}}}_3)_{\eta}=\sum_{i=1}^{2}(-1)^{3-i}\left[\frac{\partial(\vec{\tilde{\overline{H}}}_1)_{\xi_{3-i}}}{\partial\tilde{\xi}_i}+\frac{\tilde{\eta}}{\tilde{d}_i}\frac{\partial(\vec{\tilde{\overline{H}}}_0)_{\xi_{3-i}}}{\partial\tilde{\xi}_i}\right] \tag{37d}$$

$$\frac{\partial(\vec{\tilde{\overline{H}}}_3)_{\eta}}{\partial\tilde{\eta}}=(\vec{\tilde{\overline{H}}}_2)_{\eta}\sum_{i=1}^{2}\tilde{d}_i^{-1}+\tilde{\eta}(\vec{\tilde{\overline{H}}}_1)_{\eta}\sum_{i=1}^{2}\tilde{d}_i^{-2}$$
$$-\sum_{i=1}^{2}\left[\frac{\partial(\vec{\tilde{\overline{H}}}_2)_{\xi_i}}{\partial\tilde{\xi}_i}+\frac{\tilde{\eta}}{\tilde{d}_i}\frac{\partial(\vec{\tilde{\overline{H}}}_1)_{\xi_i}}{\partial\tilde{\xi}_i}+\frac{\tilde{\eta}^2}{\tilde{d}_i^2}\frac{\partial(\vec{\tilde{\overline{H}}}_0)_{\xi_i}}{\partial\tilde{\xi}_i}\right] \tag{37e}$$

The procedure described above can be continued and the equations for the following terms of expansions can be derived. However, in the present paper we restrict ourselves to four terms of the expansions ($m=3$) and neglect the terms of the order $O(p^4)$. Note that the use of higher order SIBCs for practical computations is restricted, because one has to compute (or *a priori* know) the principal curvatures at every point.

The representation (32) has clear physical meaning, namely:

(1) The zero-order terms of the expansions (32) give the solution of the problem in so-called perfect electrical conductor limit (PEC), in which the magnetic field diffusion into the body is neglected.

(2) The first-order terms describe the diffusion in the well-known Leontovich approximation, in which the body's surface is considered as a plane and the field is assumed to penetrate into the body only in the direction normal to the body's surface.

(3) The second-order terms provide a correction that takes into account the curvature of the body's surface, but the diffusion is assumed to be only in the direction normal to the surface as in the Leontovich approximation. This is Mitzner's approximation.

(4) The third-order terms and higher allow for the magnetic field diffusion in directions tangential to the body's surface. This approximation will be referred as Rytov's approximation.

Below, the problems described by (35)–(37) are solved sequentially to derive the first-, second- and third-order terms in explicit form.

6 Surface impedance boundary conditions in the Laplace domain

6.1 Leontovich's approximation

The first-order terms of the expansions of the electric and magnetic fields at the body's surface ($\eta = 0$) can be expressed directly from (35b) and (35c) as follows:

$$(\overline{\vec{E}}_1)^b_{\xi_k} = (-1)^k \left.\frac{\partial(\vec{\vec{\tilde{H}}}_0)_{\xi_{3-k}}}{\partial\tilde{\eta}}\right|_{\tilde{\eta}=0} \quad k = 1,2 \tag{38a}$$

$$(\vec{\vec{\tilde{H}}}_1)^b_{\eta} = \frac{1}{\tilde{s}}\sum_{i=1}^{2}(-1)^i \frac{\partial(\vec{\vec{\tilde{E}}}_1)^b_{\xi_{3-i}}}{\partial\tilde{\xi}_i} \tag{38b}$$

Here and below, the superscript 'b' denotes quantities on the surface of the conductor. The distribution of the functions $(\vec{\vec{\tilde{H}}}_0)_{\xi_i}$ in the direction normal to the body's surface is described by the following 1-dimensional diffusion equation obtained from (35a) and (35b):

$$\frac{\partial^2(\vec{\vec{\tilde{H}}}_0)_{\xi_k}}{\partial\tilde{\eta}^2} - \tilde{s}(\vec{\vec{\tilde{H}}}_0)_{\xi_k} = 0 \quad k = 1,2 \tag{39}$$

Equation (39) must be supplemented by the following conditions:

$$\tilde{\eta} = 0: \quad (\vec{\vec{\tilde{H}}}_0)_{\xi_k} = (\vec{\vec{\tilde{H}}}_0)^b_{\xi_k}; \quad \eta \to \infty: (\vec{\tilde{H}}_0)_{\xi_k} \to 0 \tag{40}$$

The solution of (39) and (40) is written in the form:

$$(\vec{\tilde{H}}_0)_{\xi_k} = (\vec{\tilde{H}}_0)^b_{\xi_k}\ \exp\left(-\tilde{\eta}\sqrt{\tilde{s}}\right), \quad k = 1,2 \tag{41}$$

By substituting (41) into (38a) and (38b), the functions $(\vec{\vec{\tilde{E}}}_1)^b_{\xi_k}$ and $(\vec{\tilde{H}}_1)^b_{\eta}$ are obtained:

$$(\vec{\vec{\tilde{E}}}_1)^b_{\xi_k} = (-1)^{3-k}\sqrt{\tilde{s}}(\vec{\vec{\tilde{H}}}_0)^b_{\xi_{3-k}}, \quad k = 1,2 \tag{42a}$$

$$(\vec{\vec{\tilde{H}}}_1)^b_{\eta} = \frac{1}{\sqrt{\tilde{s}}}\sum_{i=1}^{2}\frac{\partial(\vec{\vec{\tilde{H}}}_0)^b_{\xi_i}}{\partial\tilde{\xi}_i} \tag{42b}$$

Since the zero-order terms of the expansions of the functions $\overline{\vec{E}}^b_{\xi i}$ and $\overline{\vec{H}}^b_{\eta}$ are zero, the Leontovich order SIBC in the Laplace domain can be written from (42a) and (42b) as follows:

$$\begin{aligned}\overline{\vec{E}}^b_{\xi_k} &= p(-1)^{3-k}\sqrt{\tilde{s}}(\vec{\vec{\tilde{H}}}_0)^b_{\xi_{3-k}} + O(p^2)\\ &= p(-1)^{3-k}\sqrt{\tilde{s}}\overline{\vec{H}}^b_{\xi_{3-k}} + O(p^2) \quad k = 1,2\end{aligned} \tag{43a}$$

$$\begin{aligned}\overline{\vec{H}}^b_{\eta} &= \frac{p}{\sqrt{\tilde{s}}}\sum_{i=1}^{2}\frac{\partial(\vec{\vec{\tilde{H}}}_0)^b_{\xi_i}}{\partial\tilde{\xi}_i} + O(p^2)\\ &= \frac{p}{\sqrt{\tilde{s}}}\sum_{i=1}^{2}\frac{\partial(\overline{\vec{H}})^b_{\xi_i}}{\partial\tilde{\xi}_i} + O(p^2)\end{aligned} \tag{43b}$$

Note that (42a) and (42b) can also be obtained by integrating (35a) and (35d) over the boundary layer with respect to η:

$$(\vec{\vec{\tilde{E}}}_1)_{\xi_k}\Big|_{\tilde{\eta}=0}^{\tilde{\eta}=\infty} = -(\vec{\vec{\tilde{E}}}_1)^b_{\xi_k} = (-1)^k\tilde{s}\int_0^{\infty}(\vec{\vec{\tilde{H}}}_0)_{\xi_{3-k}}d\tilde{\eta} \quad k = 1,2 \tag{44a}$$

$$(\vec{\vec{\tilde{H}}}_1)_{\eta}\Big|_{\tilde{\eta}=0}^{\tilde{\eta}=\infty} = -(\vec{\vec{\tilde{H}}}_1)^b_{\eta} = -\sum_{i=1}^{2}\frac{\partial}{\partial\tilde{\xi}_i}\int_0^{\infty}(\vec{\vec{\tilde{H}}}_0)_{\xi_i}d\tilde{\eta} \tag{44b}$$

It is a simple matter to verify that substitution of (41) into (44a) and (44b) leads to (43a) and (43b).

6.2 Mitzner's approximation

The functions $(\vec{\vec{\tilde{E}}}_2)^b_{\xi_k}$ and $(\vec{\vec{\tilde{H}}}_2)^b_{\eta}$ can be derived from (36b) and (36c) as follows:

$$(\vec{\vec{\tilde{E}}}_2)^b_{\xi_k} = (-1)^k\left[\left.\frac{\partial(\vec{\vec{\tilde{H}}}_1)_{\xi_{3-k}}}{\partial\tilde{\eta}}\right|_{\tilde{\eta}=0} - \frac{(\vec{\vec{\tilde{H}}}_0)^b_{\xi_{3-k}}}{\tilde{d}_{3-k}}\right] \quad k = 1,2 \tag{45a}$$

$$(\vec{\vec{\tilde{H}}}_2)^b_{\eta} = \frac{1}{s}\sum_{i=1}^{2}(-1)^i\frac{\partial(\vec{\vec{\tilde{E}}}_2)^b_{\xi_{3-i}}}{\partial\tilde{\xi}_i} \tag{45b}$$

The diffusion equation for the functions $(\vec{\vec{\tilde{H}}}_1)_{\xi_1}$ can be obtained from (36a) and (36b) and written in the following form:

$$\begin{aligned}\frac{\partial^2(\vec{\vec{\tilde{H}}}_1)_{\xi_{3-k}}}{\partial\tilde{\eta}^2} - \tilde{s}(\vec{\vec{\tilde{H}}}_1)_{\xi_{3-k}} &= \tilde{d}^{-1}_{3-k}\frac{\partial(\vec{\vec{\tilde{H}}}_0)_{\xi_{3-k}}}{\partial\tilde{\eta}}\\ &+ (-1)^k\tilde{d}^{-1}_k(\vec{\vec{\tilde{E}}}_1)_{\xi_k}, \quad k = 1,2\end{aligned} \tag{46}$$

Note that the right-hand side of (46) is not zero unlike the diffusion equation (39) for $(\vec{\vec{\tilde{H}}}_0)_{\xi_1}$ in the Leontovich approximation. Taking into account (41) and (35b), one obtains:

$$(-1)^k(\vec{\vec{\tilde{E}}}_1)_{\xi_k} = \frac{\partial(\vec{\vec{\tilde{H}}}_0)_{\xi_{3-k}}}{\partial\tilde{\eta}} = -\sqrt{\tilde{s}}(\vec{\vec{\tilde{H}}}_0)^b_{\xi_{3-k}}\exp(-\tilde{\eta}\sqrt{\tilde{s}}) \quad k = 1,2 \tag{47}$$

Substitution of (47) into (46) yields:

$$\begin{aligned}&\frac{\partial^2(\vec{\vec{\tilde{H}}}_1)_{\xi_{3-k}}}{\partial\tilde{\eta}^2} - \tilde{s}(\vec{\vec{\tilde{H}}}_1)_{\xi_{3-k}}\\ &= -\tilde{d}_{12}\sqrt{\tilde{s}}(\vec{\vec{\tilde{H}}}_0)^b_{\xi_{3-k}}\exp(-\tilde{\eta}\sqrt{\tilde{s}}) \quad k = 1,2\end{aligned} \tag{48}$$

where $\tilde{d}_{12} = \sum_{i=1}^{2} \tilde{d}_i^{-1}$. Equation (48) must be supplemented by the following conditions:

$$\begin{aligned}\tilde{\eta} = 0 &: (\vec{\bar{\tilde{H}}}_1)_{\xi_k} = (\vec{\bar{\tilde{H}}}_1)^b_{\xi_k}(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{s});\\ \tilde{\eta} \to \infty &: (\vec{\bar{\tilde{H}}}_1)_{\xi_k} \to 0 \quad k = 1, 2\end{aligned} \tag{49}$$

The solution of (48) and (49) is written in the form

$$(\vec{\bar{\tilde{H}}}_1)_{\xi_{3-k}} = \left[(\vec{\bar{\tilde{H}}}_1)^b_{\xi_{3-k}} + \frac{\tilde{\eta}}{2}\tilde{d}_{12}(\vec{\bar{\tilde{H}}}_0)^b_{\xi_{3-k}}\right]\exp(-\tilde{\eta}\sqrt{\tilde{s}}) \quad k = 1, 2 \tag{50}$$

By substituting (50) into (45), the functions $(\vec{\bar{\tilde{E}}}_2)^b$ $(\vec{\bar{\tilde{H}}}_2)^b_\eta$ are finally obtained:

$$(\vec{\bar{\tilde{E}}}_2)^b_{\xi_k} = (-1)^{3-k}\left\{\sqrt{\tilde{s}}(\vec{\bar{\tilde{H}}}_1)^b_{\xi_{3-k}} + \frac{1}{2}\left(\tilde{d}^{-1}_{\xi_{3-k}} - \tilde{d}_k^{-1}\right)(\vec{\bar{\tilde{H}}}_0)^b_{\xi_{3-k}}\right\} \quad k = 1, 2 \tag{51a}$$

$$\begin{aligned}(\vec{\bar{\tilde{H}}}_1)^b_\eta = &\frac{1}{\sqrt{\tilde{s}}}\sum_{i=1}^{2}\frac{\partial(\vec{\bar{\tilde{H}}}_1)^b_{\xi_i}}{\partial\tilde{\xi}_i}\\ &+ \frac{1}{2\tilde{s}}\sum_{i=1}^{2}\left(\tilde{d}_i^{-1} - \tilde{d}_{3-i}^{-1}\right)\frac{\partial(\vec{\bar{\tilde{H}}}_0)^b_{\xi_i}}{\partial\tilde{\xi}_i}\end{aligned} \tag{51b}$$

Another way to obtain the formulae in (51) is by integration of (36a) and (36e) over the boundary layer as follows:

$$\begin{aligned}(\vec{\bar{\tilde{E}}}_2)^b_{\xi_k} = &-\tilde{d}_k^{-1}\int_0^\infty(\vec{\bar{\tilde{E}}}_1)_{\xi_k}d\tilde{\eta}\\ &+ (-1)^{3-k}s\int_0^\infty(\vec{\bar{\tilde{H}}}_1)_{\xi_{3-k}}d\tilde{\eta}\\ &k = 1, 2\end{aligned} \tag{52a}$$

$$\begin{aligned}(\vec{\bar{\tilde{H}}}_1)^b_\eta = &-\sum_{i=1}^{2}\tilde{d}_i^{-1}\int_0^\infty(\vec{\bar{\tilde{H}}}_1)_\eta d\tilde{\eta}\\ &+ \sum_{i=1}^{2}\frac{\partial}{\partial\tilde{\xi}_i}\int_0^\infty\left[(\vec{\bar{\tilde{H}}}_1)_{\xi_i} + \tilde{d}_i^{-1}\tilde{\eta}(\vec{\bar{\tilde{H}}}_0)_{\xi_i}\right]d\tilde{\eta}\end{aligned} \tag{52b}$$

Substitution of (52) into (45) leads to (51).

By combining (43) and (51) the Mitzner order SIBCs in the Laplace domain are written in the following form:

$$\begin{aligned}\bar{\tilde{E}}^b_{\xi_k} =&(-1)^{3-k}p\sqrt{\tilde{s}}\left\{(\vec{\bar{\tilde{H}}}_0)^b_{\xi_{3-k}} + p\left[(\vec{\bar{\tilde{H}}}_1)^b_{\xi_{3-k}}\right.\right.\\ &\left.\left.+ \frac{1}{2\sqrt{\tilde{s}}}\left(\tilde{d}_{3-k}^{-1} - \tilde{d}_k^{-1}\right)(\vec{\bar{\tilde{H}}}_0)^b_{\xi_{3-k}}\right]\right\} + O(p^3)\\ =&(-1)^{3-k}p\sqrt{\tilde{s}}\left\{1 + \frac{p}{2\sqrt{\tilde{s}}}\left(\tilde{d}_{3-k}^{-1} - \tilde{d}_k^{-1}\right)\right\}\bar{\tilde{H}}^b_{\xi_{3-k}}\\ &+ O(p^3) \quad k = 1, 2\end{aligned} \tag{53a}$$

$$\begin{aligned}\bar{\tilde{H}}^b_\eta =& \frac{p}{\sqrt{\tilde{s}}}\sum_{i=1}^{2}\frac{\partial}{\partial\tilde{\xi}_i}\left\{\left(\bar{\tilde{H}}_0\right)^b_{\xi_k}\right.\\ &\left.+ p\left[\left(\bar{\tilde{H}}_1\right)^b_{\xi_i} + \frac{1}{2\sqrt{\tilde{s}}}\left(\tilde{d}_i^{-1} - \tilde{d}_{3-i}^{-1}\right)\left(\bar{\tilde{H}}_0\right)^b_{\xi_i}\right]\right\} + O(p^3)\\ =& \frac{p}{\sqrt{\tilde{s}}}\sum_{i=1}^{2}\left(1 + \frac{p}{\sqrt{\tilde{s}}}\right)\left(\tilde{d}_i^{-1} - \tilde{d}_{3-i}^{-1}\right)\frac{\partial\bar{\tilde{H}}^b_{\xi_i}}{\partial\tilde{\xi}_i} + O(p^3)\end{aligned} \tag{53b}$$

6.3 Rytov's approximation

From (37b) and (37c) one obtains:

$$(\vec{\tilde{E}}_3)^b_{\xi_k} = (-1)^k\left[\left.\frac{\partial(\vec{\bar{\tilde{H}}}_2)_{\xi_{3-k}}}{\partial\tilde{\eta}}\right|_{\tilde{\eta}=0} - \frac{(\vec{\bar{\tilde{H}}}_1)^b_{\xi_{3-k}}}{\tilde{d}_{3-k}} - \frac{\partial(\vec{\bar{\tilde{H}}}_1)^b_\eta}{\partial\tilde{\xi}_{3-k}}\right] \quad k = 1, 2 \tag{54a}$$

$$(\vec{\bar{\tilde{H}}}_3)^b_\eta = \frac{1}{\tilde{s}}\sum_{i=1}^{2}(-1)^i\frac{\partial(\vec{\bar{\tilde{E}}}_3)^b_{\xi_{3-i}}}{\partial\tilde{\xi}_i} \tag{54b}$$

The distribution of the functions $(\vec{\bar{\tilde{H}}}_2)_{\xi_i}$ over the boundary layer is described by the following 1-dimensional problem obtained from (37a) and (37b):

$$\begin{aligned}\frac{\partial^2(\vec{\bar{\tilde{H}}}_2)_{\xi_{3-k}}}{\partial\tilde{\eta}^2} - \tilde{s}(\vec{\bar{\tilde{H}}}_2)_{\xi_{3-k}} =& \frac{(-1)^k}{\tilde{d}_k}(\vec{\bar{\tilde{E}}}_2)_{\xi_k} + \frac{(-1)^k}{\tilde{d}_k^2}\tilde{\eta}(\vec{\bar{\tilde{E}}}_1)_{\xi_k}\\ &+ (-1)^k\frac{\partial(\vec{\bar{\tilde{E}}}_2)_\eta}{\partial\tilde{\xi}_k} + \frac{1}{\tilde{d}_{3-k}}\frac{\partial(\vec{\bar{\tilde{H}}}_1)_{\xi_{3-k}}}{\partial\tilde{\eta}} + \frac{(\vec{\bar{\tilde{H}}}_0)_{\xi_{3-k}}}{\tilde{d}_{3-k}^2}\\ &+ \frac{\tilde{\eta}}{\tilde{d}_{3-k}^2}\frac{\partial(\vec{\bar{\tilde{H}}}_0)_{\xi_{3-k}}}{\partial\tilde{\eta}} + \frac{\partial^2(\vec{\bar{\tilde{H}}}_1)_\eta}{\partial\tilde{\xi}_{3-k}\partial\tilde{\eta}} \quad k = 1,\ 2\end{aligned} \tag{55a}$$

$$\begin{aligned}\tilde{\eta} = 0 &: (\vec{\bar{\tilde{H}}}_2)_{\xi_k} = (\vec{\bar{\tilde{H}}}_2)^b_{\xi_k}(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{s});\\ \tilde{\eta} \to \infty &: (\vec{\bar{\tilde{H}}}_2)_{\xi_k} \to 0\end{aligned} \tag{55b}$$

Substituting the solution of the problem in (55) into (54) and combining the result with (53), the Rytov order SIBC in the Laplace domain is written in the following form (details of this derivation are given in the Appendix):

$$\bar{\tilde{E}}^b_{\xi_k} = (-1)^{3-k}\sqrt{\tilde{s}} \times \left\{ \left(p + \frac{p^2}{\sqrt{\tilde{s}}}\frac{\tilde{d}_k - \tilde{d}_{3-k}}{2\tilde{d}_k\tilde{d}_{3-k}} + \frac{p^3}{\tilde{s}}\frac{3\tilde{d}_k^2 - \tilde{d}_{3-k}^2 - 2\tilde{d}_k\tilde{d}_{3-k}}{8\tilde{d}_k^2\tilde{d}_{3-k}^2} \right)\bar{\tilde{H}}^b_{\xi_{3-k}} + \frac{p^3}{2\tilde{s}}\left(-\frac{\partial^2 \bar{\tilde{H}}^b_{\xi_{3-k}}}{\partial\tilde{\xi}_k^2} + \frac{\partial^2 \bar{\tilde{H}}^b_{\xi_{3-k}}}{\partial\tilde{\xi}_{3-k}^2} + 2\frac{\partial^2 \bar{\tilde{H}}^b_{\xi_k}}{\partial\tilde{\xi}_{3-k}\partial\tilde{\xi}_k} \right) \right\} + O(p^4) \quad k = 1,2 \tag{56a}$$

$$\bar{\tilde{H}}^b_\eta = \frac{2}{\sqrt{\tilde{s}}}\sum_{i=1}^{2}\frac{\partial}{\partial\tilde{\xi}_i}\left\{ \left(p + \frac{p^2}{\sqrt{\tilde{s}}}\frac{\tilde{d}_{3-i} - \tilde{d}_i}{2\tilde{d}_i\tilde{d}_{3-i}} + \frac{p^2}{\tilde{s}}\frac{3\tilde{d}_{3-i}^2 - \tilde{d}_i^2 - 2\tilde{d}_i\tilde{d}_{3-i}}{8\tilde{d}_i^2\tilde{d}_{3-i}^2} \right)\bar{\tilde{H}}^b_{\xi_i} + \frac{p^2}{2\tilde{s}}\left(-\frac{\partial^2 \bar{\tilde{H}}^b_{\xi_i}}{\partial\tilde{\xi}_{3-i}^2} + \frac{\partial^2 \bar{\tilde{H}}^b_{\xi_i}}{\partial\tilde{\xi}_i^2} + 2\frac{\partial^2 \bar{\tilde{H}}^b_{\xi_{3-i}}}{\partial\tilde{\xi}_i\partial\tilde{\xi}_{3-i}} \right) \right\} + O(p^4) \tag{56b}$$

The conditions (56) can also be represented in the following form:

$$\bar{\tilde{E}}^b_{\xi_k} = (-1)^{3-k}\tilde{s}\bar{\tilde{F}}_{3-k} \quad k = 1,2 \tag{57a}$$

$$\bar{\tilde{H}}^b_\eta = \sum_{i=1}^{2}\frac{\partial\bar{\tilde{F}}_i}{\partial\tilde{\xi}_i} \tag{57b}$$

where the Laplace-domain functions $\bar{\tilde{F}}_k = \bar{\tilde{F}}_k(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{s}), k = 1, 2$, are as follows

$$\bar{\tilde{F}}_{3-k} = \frac{p}{\sqrt{\tilde{s}}}\bar{\tilde{H}}^b_{\xi_{3-k}} + \frac{p^2}{\tilde{s}}\frac{\tilde{d}_k - \tilde{d}_{3-k}}{2\tilde{d}_k\tilde{d}_{3-k}}\bar{\tilde{H}}^b_{\xi_{3-k}} + \frac{p^3}{\tilde{s}}\frac{3\tilde{d}_k^2 - \tilde{d}_{3-k}^2 - 2\tilde{d}_k\tilde{d}_{3-k}}{8\tilde{d}_k^2\tilde{d}_{3-k}^2}\bar{\tilde{H}}^b_{\xi_{3-k}} + \frac{p^3}{2\tilde{s}^{3/2}}\left(-\frac{\partial^2 \bar{\tilde{H}}^b_{\xi_{3-k}}}{\partial\tilde{\xi}_k^2} + \frac{\partial^2 \bar{\tilde{H}}^b_{\xi_{3-k}}}{\partial\tilde{\xi}_{3-k}^2} + 2\frac{\partial^2 \bar{\tilde{H}}^b_{\xi_k}}{\partial\tilde{\xi}_k\partial\tilde{\xi}_{3-k}} \right) + O(p^4) \tag{57c}$$

7 Surface impedance boundary conditions in the time domain

Using the inverse Laplace transform, the following time-domain functions T_m are derived [22]:

$$\frac{1}{\sqrt{\tilde{s}}} \Leftrightarrow \frac{1}{\sqrt{\pi\tilde{t}}} = \tilde{T}_1(\tilde{t}); \quad \frac{1}{\tilde{s}} \Leftrightarrow U(\tilde{t}) = \tilde{T}_2(\tilde{t}); \quad \frac{1}{\tilde{s}^{3/2}} \Leftrightarrow \frac{2}{\sqrt{\pi}}\sqrt{\tilde{t}} = \tilde{T}_3(\tilde{t}) \tag{58}$$

where $U(t)$ is the unit step function. Substituting (58) into (57) and applying the Duhamel theorem, one obtains:

$$\tilde{E}^b_{\xi_k} = (-1)^{3-k}\frac{\partial\tilde{F}_{3-k}}{\partial t} \tag{59a}$$

$$\tilde{H}^b_\eta = \sum_{i=1}^{2}\frac{\partial\tilde{F}_i}{\partial\tilde{\xi}_i} \tag{59b}$$

$$\tilde{F}_{3-k} = p\tilde{T}_1\tilde{*}\tilde{H}^b_{\xi_{3-k}} + p^2\frac{\tilde{d}_k - \tilde{d}_{3-k}}{2\tilde{d}_k\tilde{d}_{3-k}}\tilde{T}_2\tilde{*}\tilde{H}^b_{\xi_{3-k}} + p^3\frac{3\tilde{d}_k^2 - \tilde{d}_{3-k}^2 - 2\tilde{d}_k\tilde{d}_{3-k}}{8\tilde{d}_k^2\tilde{d}_{3-k}^2}\tilde{T}_3\tilde{*}\tilde{H}^b_{\xi_{3-k}} + \frac{p^3}{2}\tilde{T}_3\tilde{*}\left(-\frac{\partial^2 \tilde{H}^b_{\xi_{3-k}}}{\partial\tilde{\xi}_k^2} + \frac{\partial^2 \tilde{H}^b_{\xi_{3-k}}}{\partial\tilde{\xi}_{3-k}^2} + 2\frac{\partial^2 \tilde{H}^b_{\xi_k}}{\partial\tilde{\xi}_k\partial\tilde{\xi}_{3-k}} \right) + O(p^4) \quad k = 1,2 \tag{59c}$$

where the sign '$\tilde{*}$' denotes a time convolution product with respect to non-dimensional time.

The functions F_k describe the perturbation of the external field surrounding the body owing to the field diffusion into the body and dissipation of the energy by the body. The first term on the right-hand side of (59c) gives the contribution from the field diffusion in the direction normal to the planar surface. The second and third terms give a correction due to the curvature. The fourth term takes into account the field diffusion in the directions tangential to the planar surface. The functions T_m describe the evolution of these processes in time.

The SIBC for the tangential electric field can be represented in a more traditional form than (59a):

$$\tilde{E}^b_{\xi_k} = (-1)^{3-k}p\left\{ \tilde{\tilde{T}}_1\tilde{*}\tilde{H}^b_{\xi_{3-k}} + p\frac{\tilde{d}_k - \tilde{d}_{3-k}}{2\tilde{d}_k\tilde{d}_{3-k}}\tilde{\tilde{T}}_2\tilde{*}\tilde{H}^b_{\xi_{3-k}} + p^2\frac{3\tilde{d}_k^2 - \tilde{d}_{3-k}^2 - 2\tilde{d}_k\tilde{d}_{3-k}}{8\tilde{d}_k^2\tilde{d}_{3-k}^2}\tilde{\tilde{T}}_3\tilde{*}\tilde{H}^b_{\xi_{3-k}} + \frac{p^2}{2}\tilde{\tilde{T}}_3\tilde{*}\left(-\frac{\partial^2 \tilde{H}^b_{\xi_{3-k}}}{\partial\tilde{\xi}_k^2} + \frac{\partial^2 \tilde{H}^b_{\xi_{3-k}}}{\partial\tilde{\xi}_{3-k}^2} + 2\frac{\partial^2 \tilde{H}^b_{\xi_k}}{\partial\tilde{\xi}_k\partial\tilde{\xi}_{3-k}} \right) \right\} + O(p^4) \quad k = 1,2 \tag{59d}$$

where

$$\tilde{\tilde{T}}_1(\tilde{t}) = -(4\pi)^{-1/2}\tilde{t}^{-3/2}; \quad \tilde{\tilde{T}}_2(\tilde{t}) = U'(\tilde{t}); \quad \tilde{\tilde{T}}_3(\tilde{t}) = \pi^{-1/2}\tilde{t}^{-1/2} \tag{60}$$

where $U'(\tilde{t})$ is the delta-function.

Table 1: Quantities and appropriate scale factors

Quantity	Scale factor	Unit
$\partial/\partial\xi_1, \partial/\partial\xi_2$	D^{-1}	m^{-1}
E	$(4\pi)^{-1}I^*\tau^{-1}$	Vm^{-1}
H	$(4\pi)^{-1}I^*D^{-1}$	Am^{-1}
T_k, k= 1,2,3	$(\sigma\mu_0)^{-k/2}\tau^{(k-1)/2} = p^k\tau^{-1}D^k$	m^ks^{-1}
dt (in time convolution product)	τ	s

Returning to dimensional variables in (59) gives (see Table 1):

$$
\begin{aligned}
E^b_{\xi_k} =&(-1)^{3-k}\Bigg\{\hat{T}_1 * H^b_{\xi_{3-k}} + \frac{d_k - d_{3-k}}{2d_k d_{3-k}}\hat{T}_2 * H^b_{\xi_{3-k}}. \\
&+\frac{3d_k^2 - d_{3-k}^2 - 2d_k d_{3-k}}{8d_k^2 d_{3-k}^2}\hat{T}_3 * H^b_{\xi_{3-k}} \\
&+\frac{\hat{T}_3}{2} * \left(-\frac{\partial^2 H^b_{\xi_{3-k}}}{\partial \xi_k^2} + \frac{\partial^2 H^b_{\xi_{3-k}}}{\partial \xi_{3-k}^2} + 2\frac{\partial^2 H^b_{\xi_k}}{\partial \xi_k \partial \xi_{3-k}}\right)\Bigg\} + \ldots \\
&k = 1, 2
\end{aligned}
\tag{61a}
$$

$$
\begin{aligned}
H^b_\eta =&\sum_{i-1}^{2}\frac{\partial}{\partial \xi_i}\Bigg\{T_1 * H^b_{\xi_i} + \frac{d_{3-i} - d_i}{2d_i d_{3-i}}T_2 * H^b_{\xi_i}. \\
&+\frac{3d_{3-i}^2 - d_i^2 - 2d_i d_{3-i}}{8d_i^2 d_{3-i}^2}T_3 * H^b_{\xi_i} \\
&+\frac{T_3}{2} * \left(\frac{\partial^2 H^b_{\xi_i}}{\partial \xi_{3-i}^2} + \frac{\partial^2 H^b_{\xi_i}}{\partial \xi_i^2} + 2\frac{\partial^2 H^b_{\xi_{3-i}}}{\partial \xi_i \partial \xi_{3-i}}\right)\Bigg\} \\
&+\ldots
\end{aligned}
\tag{61b}
$$

where $T_1(t) = (\pi\sigma\mu)^{-1/2}t^{-1/2}$; $T_2(t) = U(t)/(\sigma\mu)$; $T_3(t) = 2(\pi\sigma^3\mu^3)^{-1/2}t^{-1/2}$; $\hat{T}_1(t) = -(4\pi\sigma/\mu)^{-1/2}t^{-3/2}$; $\hat{T}_2(t) = U'(t)/\sigma$; and $\hat{T}_3(t) = (\pi\sigma^3\mu)^{-1/2}t^{-1/2}$.

The time domain surface impedance boundary conditions in (61) are the first main result of this work.

As a conclusion of this Section let us demonstrate how the original frequency domain Mitzner's and Rytov's conditions can be obtained from (56). Since the ratio $2/\omega$ is the time scale factor in the time harmonic case, all we have to do is to replace $\tilde{s}$ by $2j$ as follows:

$$
\begin{aligned}
\tilde{\dot{E}}^b_{\xi_k} =&(-1)^{3-k}p(1+j)\Bigg\{\Bigg[1 + p\frac{1-j}{4}\left(\tilde{d}_{3-k}^{-1} - \tilde{d}_k^{-1}\right). \\
&+\frac{p^2}{2j}\frac{3\tilde{d}_k^2 - \tilde{d}_{3-k}^2 - 2\tilde{d}_k\tilde{d}_{3-k}}{8\tilde{d}_k^2\tilde{d}_{3-k}^2}\Bigg]\tilde{\dot{H}}^b_{\xi_{3-k}} \\
&+\frac{p^3}{2j}\left(-\frac{\partial^2 \tilde{\dot{H}}^b_{\xi_{3-k}}}{\partial \tilde{\xi}_k^2} + \frac{\partial^2 \tilde{\dot{H}}^b_{\xi_{3-k}}}{\partial \tilde{\xi}_{3-k}^2} + 2\frac{\partial^2 \tilde{\dot{H}}^b_{\xi_k}}{\partial \tilde{\xi}_k \partial \tilde{\xi}_{3-k}}\right)\Bigg\} \\
&+O(p^4) \quad k = 1, 2
\end{aligned}
\tag{62}
$$

where the sign '·' denotes the amplitude of the function. Substitution of the scale factors gives the condition in dimensional form

$$
\begin{aligned}
\dot{E}^b_{\xi_k} =&(-1)^{3-k}(1+j)Z_\omega\Bigg\{\Bigg[1 + \frac{1-j}{4}\delta\left(d_{3-k}^{-1} - d_k^{-1}\right). \\
&+\frac{\delta^2}{2j}\frac{3d_k^2 - d_{3-k}^2 - 2d_k d_{3-k}}{8d_k^2 d_{3-k}^2}\Bigg]\dot{H}^b_{\xi_{3-k}} \\
&+\frac{\delta^2}{4j}\left(-\frac{\partial^2 \dot{H}^b_{\xi_{3-k}}}{\partial \xi_k^2} + \frac{\partial^2 \dot{H}^b_{\xi_{3-k}}}{\partial \xi_{3-k}^2} + 2\frac{\partial^2 \dot{H}^b_{\xi_k}}{\partial \xi_k \partial \xi_{3-k}}\right)\Bigg\} \\
&+\ldots \quad k = 1, 2
\end{aligned}
\tag{63}
$$

It is easy to see that by neglecting the terms of the order $O(p^3)$, the formula (63) reduces to the Mitzner condition [8]. Alternatively, by setting $d_1 \to \infty$, $d_2 \to \infty$ in (63) the Rytov condition [7] is obtained.

8 Conclusions

The problem of the diffusion of the transient electromagnetic field in a conductor has been solved using the method of perturbations in the small parameter p that is equal to the ratio of the electromagnetic penetration depth and characteristic dimension of the body. The time domain solutions for the tangential component of the electric field and the normal component of the magnetic field on the smooth curved surface of the body (the surface impedance boundary conditions) have been obtained with accuracy up to $O(p^4)$. It was shown that the proposed SIBC in the frequency domain generalises the well-known Leontovich boundary condition and Mitzner's boundary condition, which provide approximation accuracy within the errors $O(p^2)$ and $O(p^3)$, respectively. In Part II of this paper, we demonstrate the formulation of the SIBCs developed here in conjunction with a boundary element method and apply it to the problem of transient skin and proximity effect problems in cylindrical conductors.

9 References

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10 Appendix

10.1 Calculation of the Rytov order terms of expansions of the tangential electric and normal magnetic fields at the conductor's surface

1. From (36b), (41) and (48) we obtain

$$(-1)^k \frac{(\vec{\tilde{E}}_2)_{\xi_k}}{\tilde{d}_k} = \frac{1}{\tilde{d}_k}\left[\frac{\partial(\vec{\tilde{H}}_1)_{\xi_{3-k}}}{\partial\tilde{\eta}} - \frac{(\vec{\tilde{H}}_0)_{\xi_{3-k}}}{\tilde{d}_{3-k}}\right] = \frac{1}{\tilde{d}_k}\left[-(\vec{\tilde{H}}_1)^h_{\xi_{3-k}}\sqrt{\tilde{s}} + \frac{\tilde{d}_{12}}{2}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}} - \tilde{\eta}\frac{\tilde{d}_{12}}{2}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}\sqrt{\tilde{s}} - \frac{(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}}{\tilde{d}_{3-k}}\right]\exp\left(-\tilde{\eta}\sqrt{\tilde{s}}\right) \tag{64}$$

where $\tilde{d}_{12} = \sum_{i=1}^{2} \tilde{d}_i^{-1}$.

2. From (35b) and (41),

$$\frac{(-1)^k}{\tilde{d}_k^2}\tilde{\eta}(\vec{\tilde{E}}_1)_{\xi_k} = \frac{\tilde{\eta}}{\tilde{d}_k^2}\frac{\partial(\vec{\tilde{H}}_0)_{\xi_{3-k}}}{\partial\tilde{\eta}} = -\frac{\tilde{\eta}}{\tilde{d}_k^2}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}\sqrt{\tilde{s}}\exp\left(-\tilde{\eta}\sqrt{\tilde{s}}\right) \tag{65}$$

3. From (36d) and (41),

$$(-1)^k\frac{\partial(\vec{\tilde{E}}_2)_\eta}{\partial\tilde{\xi}_k} = \sum_{i=1}^{2}(-1)^{3-i+k}\frac{\partial^2(\vec{\tilde{H}}_0)_{\xi_{3-i}}}{\partial\tilde{\xi}_k\partial\tilde{\xi}_i} = \sum_{i=1}^{2}(-1)^{3-i+k}\frac{\partial^2(\vec{\tilde{H}}_0)_{\xi_{3-i}}}{\partial\tilde{\xi}_k\partial\tilde{\xi}_i}\exp\left(-\tilde{\eta}\sqrt{\tilde{s}}\right) \tag{66}$$

4. From (35d) and (41),

$$\frac{\partial^2(\vec{\tilde{H}}_1)_\eta}{\partial\tilde{\xi}_{3-k}\partial\tilde{\eta}} = -\sum_{i=1}^{2}\frac{\partial^2(\vec{\tilde{H}}_0)_{\xi_i}}{\partial\tilde{\xi}_{3-k}\tilde{\xi}_i} = -\sum_{i=1}^{2}\frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_i}}{\partial\tilde{\xi}_{3-k}\tilde{\xi}_i}\exp\left(-\tilde{\eta}\sqrt{\tilde{s}}\right) \tag{67}$$

By using (64)–(67), (55a) can be represented in the following form:

$$\frac{\partial^2(\vec{\tilde{H}}_2)_{\xi_{3-k}}}{\partial\tilde{\eta}^2} - \tilde{s}(\vec{\tilde{H}}_2)_{\xi_{3-k}} = (\tilde{G}_{3-k} + \tilde{\eta}\tilde{W}_{3-k})\exp\left(-\tilde{\eta}\sqrt{\tilde{s}}\right) \tag{68}$$

where the functions $\tilde{G}_{3-k}$ and $\tilde{W}_{3-k}$, $k=1, 2$, are defined as follows

$$\begin{aligned}\tilde{G}_{3-k} &= -\tilde{d}_{12}(\vec{\tilde{H}}_1)^s_{\xi_{3-k}}\sqrt{\tilde{s}} + \frac{\tilde{d}_{12}^2}{2}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}} + \frac{(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}}{\tilde{d}_{3-k}}\left[\frac{1}{\tilde{d}_{3-k}} - \frac{1}{\tilde{d}_k}\right] + \sum_{i=1}^{2}\left[(-1)^{3-i+k}\frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_{3-i}}}{\partial\tilde{\xi}_k\partial\tilde{\xi}_i} - \frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_i}}{\partial\tilde{\xi}_{3-k}\partial\tilde{\xi}_i}\right]\\ &= -\tilde{d}_{12}(\vec{\tilde{H}}_1)^b_{\xi_{3-k}}\sqrt{\tilde{s}} + \frac{3\tilde{d}_k^2 + \tilde{d}_{3-k}^2}{2\tilde{d}_k^2\tilde{d}_{3-k}^2}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}} + \left[-\frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}}{\partial\tilde{\xi}_k^2} + \frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}}{\partial\tilde{\xi}_{3-k}^2} + 2\frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_k}}{\partial\tilde{\xi}_{3-k}\partial\tilde{\xi}_k}\right]\end{aligned} \tag{69a}$$

$$\tilde{W}_{3-k} = -\sqrt{\tilde{s}}\frac{3\tilde{d}_k^2 + 3\tilde{d}_{3-k}^2 + 2\tilde{d}_k\tilde{d}_{3-k}}{2\tilde{d}_k^2\tilde{d}_{3-k}^2}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}} \tag{69b}$$

The solution of equation (68) with the conditions (55b) is written in the form:

$$(\vec{\tilde{H}}_2)_{\xi_{3-k}} = \left\{(\vec{\tilde{H}}_2)^b_{\xi_{3-k}} - \frac{1}{4\sqrt{\tilde{s}}}\left[\tilde{\eta}^2\tilde{W}_{3-k} + \tilde{\eta}\left(\frac{\tilde{W}_{3-k}}{\sqrt{\tilde{s}}} + 2\tilde{G}_{3-k}\right)\right]\right\}\times\exp\left(-\tilde{\eta}\sqrt{\tilde{s}}\right) \tag{70}$$

Substitution of (69) into (70) leads to the following results:

$$\begin{aligned}(\vec{\tilde{H}}_2)_{\xi_{3-k}} = \Bigg\{&(\vec{\tilde{H}}_2)^b_{\xi_{3-k}} + \tilde{\eta}\frac{\tilde{d}_{12}}{2}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}.\\ &+\tilde{\eta}^2\frac{3\tilde{d}_k^2 + 3\tilde{d}_{3-k}^2 + 2\tilde{d}_k\tilde{d}_{3-k}}{8\tilde{d}_k^2\tilde{d}_{3-k}^2}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}\\ &+\tilde{\eta}\frac{-3\tilde{d}_k^2 + \tilde{d}_{3-k}^2 + 2\tilde{d}_k\tilde{d}_{3-k}}{8\tilde{d}_k^2\tilde{d}_{3-k}^2\sqrt{\tilde{s}}}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}\\ &-\frac{\tilde{\eta}}{2\sqrt{\tilde{s}}}\left[-\frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}}{\partial\tilde{\xi}_k^2} + \frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}}{\partial\tilde{\xi}_{3-k}^2} + 2\frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_k}}{\partial\tilde{\xi}_{3-k}\partial\tilde{\xi}_k}\right]\Bigg\}\\ &\times\exp\left(-\tilde{\eta}\sqrt{\tilde{s}}\right)\end{aligned} \tag{71}$$

$$\begin{aligned}\left.\frac{\partial(\vec{\tilde{H}}_2)_{\xi_{3-k}}}{\partial\tilde{\eta}}\right|_{\tilde{\eta}=0} &= -(\vec{\tilde{H}}_2)^b_{\xi_{3-k}}\sqrt{\tilde{s}} + \frac{\tilde{d}_{12}}{2}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}} + \frac{1}{\sqrt{\tilde{s}}}\frac{-3\tilde{d}_k^2 + \tilde{d}_{3-k}^2 + 2\tilde{d}_k\tilde{d}_{3-k}}{8\tilde{d}_k^2\tilde{d}_{3-k}^2}(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}\\ &-\frac{1}{2\sqrt{\tilde{s}}}\left[-\frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}}{\partial\tilde{\xi}_k^2} + \frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_{3-k}}}{\partial\tilde{\xi}_{3-k}^2} + 2\frac{\partial^2(\vec{\tilde{H}}_0)^b_{\xi_k}}{\partial\tilde{\xi}_{3-k}\partial\tilde{\xi}_k}\right]\end{aligned} \tag{72}$$

By substituting (72) into (54) and taking into account (42b), the desired result is obtained and written in the form:

$$(\tilde{\overline{\overline{E}}}_3)^b_{\xi_k} = (-1)^k \left\{ -\sqrt{\tilde{s}}(\tilde{\overline{\overline{H}}}_2)^b_{\xi_{3-k}} + (\tilde{\overline{\overline{H}}}_1)^b_{\xi_{3-k}} \frac{\tilde{d}_{3-k} - \tilde{d}_k}{\tilde{d}_k \tilde{d}_{3-k}} . \right.$$

$$+ \frac{(\tilde{\overline{\overline{H}}}_0)^b_{\xi_{3-k}}}{\sqrt{\tilde{s}}} \frac{-3\tilde{d}_k^2 + \tilde{d}_{3-k}^2 + 2\tilde{d}_k \tilde{d}_{3-k}}{8\tilde{d}_k^2 \tilde{d}_{3-k}^2} -$$

$$\left. - \frac{1}{2\sqrt{\tilde{s}}} \left[-\frac{\partial^2 (\tilde{\overline{\overline{H}}}_0)^b_{\xi_{3-k}}}{\partial \tilde{\xi}_k^2} + \frac{\partial^2 (\tilde{\overline{\overline{H}}}_0)^b_{\xi_{3-k}}}{\partial \tilde{\xi}_{3-k}^2} + 2 \frac{\partial^2 (\tilde{\overline{\overline{H}}}_0)^b_{\xi_k}}{\partial \tilde{\xi}_{3-k} \partial \tilde{\xi}_k} \right] \right\} \quad (73)$$

$$(\tilde{\overline{\overline{H}}}_3)^b_\eta = \frac{1}{\sqrt{\tilde{s}}} \sum_{i=1}^{2} \frac{\partial (\tilde{\overline{\overline{H}}}_2)^h_{\xi_i}}{\partial \tilde{\xi}_i} - \frac{1}{2\tilde{s}} \sum_{i=1}^{2} \frac{\tilde{d}_i - \tilde{d}_{3-i}}{\tilde{d}_i \tilde{d}_{3-i}} \frac{\partial (\tilde{\overline{\overline{H}}}_1)^b_{\xi_i}}{\partial \tilde{\xi}_i}$$

$$+ \frac{1}{\tilde{s}^{3/2}} \sum_{i=1}^{2} \frac{3\tilde{d}_{3-i}^2 - \tilde{d}_i^2 - 2\tilde{d}_i \tilde{d}_{3-i}}{8\tilde{d}_i^2 \tilde{d}_{3-i}^2} \frac{\partial (\tilde{\overline{\overline{H}}}_0)^b_{\xi_i}}{\partial \tilde{\xi}_i^2}$$

$$+ \frac{1}{2\tilde{s}^{3/2}} \sum_{i=1}^{2} \frac{\partial}{\partial \tilde{\xi}_i} \left[-\frac{\partial^2 (\tilde{\overline{\overline{H}}}_0)^b_{\xi_i}}{\partial \tilde{\xi}_{3-i}^2} + \frac{\partial^2 (\tilde{\overline{\overline{H}}}_0)^b_{\xi_i}}{\partial \tilde{\xi}_i^2} + 2 \frac{\partial^2 (\tilde{\overline{\overline{H}}}_0)^b_{\xi_{3-i}}}{\partial \tilde{\xi}_i \partial \tilde{\xi}_{3-i}} \right] \quad (74)$$