ELECTROMAGNETICS II
March 16, 2007
Exam 2

Answer the following four questions. Write clearly, concisely and legibly. You may use any material, including your notes, book, etc. but you cannot borrow material during the exam. You are encouraged to make <u>reasonable</u> assumptions but you must state them clearly.

1. An FM modulated signal propagates in a medium with properties μ_0, ε_0, $\sigma = 10^{-2}$ S/m. The frequency of the signal is f_0 = 1 MHz and the modulating signal is 1 kHz. This results in a composite signal which varies from a low frequency of 999 kHz to 1001 kHz. Calculate the phase difference between the two extreme frequencies after the the signal propagated a distance d = 1 km.

Solution: There are two ways of evaluating this problem. One is to calculate the group velocities. The simpler method is to calculate the phase constants at the two extreme frequencies and from that to calculate the phase difference. First we need to decide if this is a low or high loss medium:

$$\frac{\sigma}{\omega\varepsilon} = \frac{0.01}{2\pi\times10^{6}\times8.854\times10^{-12}} = 179.75 >> 1$$

This may be considered a high loss medium.
The phase constant for a high loss medium is:

$$\beta = \sqrt{\pi f\mu_0\sigma} = \sqrt{\pi\times f\times4\pi\times10^{-7}\times0.01} = 6.3246\pi\times10^{-5}\sqrt{f} \qquad \left[\frac{\text{rad}}{\text{m}}\right]$$

At the lowest frequency:

$$\beta_l = 6.3246\pi10^{-5}\sqrt{999{,}000} = 0.19859 \qquad \left[\frac{\text{rad}}{\text{m}}\right]$$

At the highest frequency:

$$\beta_h = 6.3246\pi\times10^{-5}\sqrt{1{,}001{,}000} = 0.19879 \qquad \left[\frac{\text{rad}}{\text{m}}\right]$$

The change in phase is:

$$\Delta\phi = (\beta_h - \beta_l)l = (0.19879 - 0.19859)\times1000 = 0.2 \qquad [\text{rad}]$$

or about 11.5º.

2. A plane wave propagates in free space and impinges on a good conductor at perpendicular incidence. The amplitude of the magnetic field intensity of the plane wave is 0.1 A/m. Calculate how deep the wave propagates in the conductor before its peak power density (not time averaged!!) is reduced to 1 μW/m^2. The properties of the conductor are: μ_0, ε_0, $\sigma = 10^7$ S/m. Assume the conductor is a high loss medium and the frequency of propagation is 1 MHz.

Solution: First we calculate the electric field intensity and then the transmission coefficient. Then we calculate the transmitted field and the Poynting vector. Since we are only interested in the peak value, the phase due to transmission or propagation is not important.

The electric field intensity is:

$$E = \eta_0 H = 377 \times 0.1 = 37.7 \qquad \left[\frac{\text{V}}{\text{m}}\right]$$

The intrinsic impedance of the conductor is:

$$\eta_c = \sqrt{\frac{j\omega\mu_0}{\sigma + j\omega\varepsilon_0}} \approx \sqrt{\frac{j\omega\mu_0}{\sigma}} = \frac{1+j}{\sqrt{2}}\sqrt{\frac{\omega\mu_0}{\sigma}} = \frac{1+j}{\sqrt{2}}\sqrt{\frac{2\pi\times 10^6 \times 4\pi\times 10^{-7}}{10^7}} = (1+j)6.28\times 10^{-4} \qquad [\Omega]$$

The transmission coefficient is:

$$T = \frac{2\eta_c}{\eta_c + \eta_0} = \frac{2\times(1+j)6.28\times 10^{-4}}{(1+j)6.28\times 10^{-4} + 377} \approx \frac{2\times(1+j)6.28\times 10^{-4}}{377} = (1+j)3.33\times 10^{-6}$$

Now we can calculate the electric field intensity transmitted across the interface. This will be called E_2:

$$E_2 = ET = 37.7(1+j)3.33\times 10^{-6} = (1+j)1.255\times 10^{-4} \qquad \left[\frac{\text{V}}{\text{m}}\right]$$

The instantaneous power density (not time averaged) is:

$$P_2 = |\mathbf{E}_2 \times \mathbf{H}_2| = \frac{E_2^2}{\eta_2} = \frac{((1+j)1.255\times 10^{-4})^2}{(1+j)6.28\times 10^{-4}} = (1+j)2.51\times 10^{-5} \qquad \left[\frac{\text{W}}{\text{m}^2}\right]$$

Now, the amplitude of this power density is

$$P_{peak} = 2.51\times 10^{-5}\sqrt{2} = 3.55\times 10^{-5} \qquad \left[\frac{\text{W}}{\text{m}^2}\right]$$

To calculate the power density as it attenuates, we need the attenuation constant. For a high loss medium:

$$\alpha = \sqrt{\pi f \mu\sigma} = \sqrt{\pi\times 10^6 \times 4\pi\times 10^{-7} \times 10^7} = 6{,}283 \qquad \left[\frac{\text{Np}}{\text{m}}\right]$$

Assuming the wave propagates a distance x before its power density is attenuated to 1 μW/m^2, we write:

$$P(x) = P(0)e^{-2\alpha x} \qquad \rightarrow \qquad 10^{-6} = 3.55\times10^{-5}e^{-2\times6283x}$$

or:

$$-2\times6283x\ln e = \ln\left(\frac{10^{-6}}{3.55\times10^{-5}}\right) \qquad \rightarrow \qquad -2\times6283x = -3.57$$

This gives:

$$x = 0.000284 \qquad [\text{m}]$$

or: 0.284 mm.

3. A plane wave propagates in free space and impinges on a conducting surface at a 45° angle. The electric field intensity has three components in space: $E_x = -10$ V/m, $E_y = 10$ V/m and $E_z = 10$ V/m. Assume the surface of the conductor coincides with the x-y plane. Calculate the time averaged power density (power per unit area) in the wave (magnitude and direction).
Hint: use superposition of plane waves.

Solution: The configuration is shown in **Figure A**. This can be seen as a superposition of a perpendicularly polarized plane wave, with electric field in the positive y direction and a parallel polarized wave with electric field components in the positive z and negative x directions. These two waves are shown, together with the magnetic fields in Figures B and C. Note that in these figures I have taken into account the fact the the reflection coefficient at the surface of a conductor is negative and therefore the tangential component of the reflected electric field intensity must flip.
We start with the perpendicularly polarized wave in **Figure B**. The electric and magnetic field intensities are as follows:

$$\mathbf{E}_\perp = \hat{\mathbf{y}}10e^{-j\beta(x\sin45^\circ + z\cos45^\circ)} - \hat{\mathbf{y}}10e^{-j\beta(x\sin45^\circ - z\cos45^\circ)} = \hat{\mathbf{y}}10\left(e^{-j\beta(x\sin45^\circ + z\cos45^\circ)} - e^{-j\beta(x\sin45^\circ - z\cos45^\circ)}\right)$$
$$= -\hat{\mathbf{y}}10\left(e^{j\beta z\cos45^\circ} - e^{-j\beta z\cos45^\circ}\right)e^{-j\beta x\sin45^\circ} = -\hat{\mathbf{y}}j20\sin(\beta z\cos45^\circ)e^{-j\beta x\sin45^\circ}$$

Note: the second term is the reflected field in Figure B. The magnetic field is:

$$\mathbf{H}_\perp = \left(-\hat{\mathbf{x}}\frac{10}{377}\cos45^\circ e^{-j\beta(x\sin45^\circ + z\cos45^\circ)} + \hat{\mathbf{z}}\frac{10}{377}\sin45^\circ e^{-j\beta(x\sin45^\circ + z\cos45^\circ)}\right) +$$
$$\left(-\hat{\mathbf{x}}\frac{10}{377}\cos45^\circ e^{-j\beta(x\sin45^\circ - z\cos45^\circ)} - \hat{\mathbf{z}}\frac{10}{377}\sin45^\circ e^{-j\beta(x\sin45^\circ - z\cos45^\circ)}\right)$$
$$= -\hat{\mathbf{x}}\frac{10}{377}\cos45^\circ\left(e^{-j\beta(x\sin45^\circ + z\cos45^\circ)} + e^{-j\beta(x\sin45^\circ - z\cos45^\circ)}\right) + \hat{\mathbf{z}}\frac{10}{377}\sin45^\circ\left(e^{-j\beta(x\sin45^\circ + z\cos45^\circ)} - e^{-j\beta(x\sin45^\circ - z\cos45^\circ)}\right)$$
$$= -\hat{\mathbf{x}}\frac{20}{377}\cos45^\circ\cos(\beta z\cos45^\circ)e^{-j\beta x\sin45^\circ} - \hat{\mathbf{z}}j\frac{20}{377}\sin45^\circ\sin(\beta z\cos45^\circ)e^{-j\beta x\sin45^\circ}$$

We can now calculate the time averaged power density for the perpendicularly polarized part of the wave:

$$\mathcal{P}_{av}^{\perp} = Re\left\{\frac{\mathbf{E}_{\perp}\times\mathbf{H}_{\perp}^{*}}{2}\right\} =$$

$$Re\left\{\frac{\left(-\hat{\mathbf{y}}j20sin(\beta zcos45°)e^{-j\beta xsin45°}\right)\times\left(-\hat{\mathbf{x}}\frac{20}{377}cos45°cos(\beta zcos45°)e^{+j\beta xsin45°} + \hat{\mathbf{z}}j\frac{20}{377}sin45°sin(\beta zcos45°)e^{+j\beta xsin45°}\right)}{2}\right\}$$

$$= \hat{\mathbf{x}}\frac{200}{377}sin^2(\beta zcos45°) = \hat{\mathbf{x}}0.53sin^2(\beta zcos45°) \qquad \left[\frac{\text{W}}{\text{m}^2}\right]$$

Note that real power propagates along the conductor in the positive x direction as expected.

Now we repeat the process for the parallel polarized wave using Figure C. The fields are:

$$\mathbf{E}_{\parallel} = \left(-\hat{\mathbf{x}}10e^{-j\beta(xsin45° + zcos45°)} + \hat{\mathbf{z}}10e^{-j\beta(xsin45° + zcos45°)}\right) +$$
$$\left(+\hat{\mathbf{x}}10e^{-j\beta(xsin45° - zcos45°)} + \hat{\mathbf{z}}10e^{-j\beta(xsin45° - zcos45°)}\right)$$
$$= \hat{\mathbf{x}}10\left(e^{-j\beta(xsin45° - zcos45°)} - e^{-j\beta(xsin45° + zcos45°)}\right) + \hat{\mathbf{z}}10\left(e^{-j\beta(xsin45° + zcos45°)} + e^{-j\beta(xsin45° - zcos45°)}\right)$$
$$= \hat{\mathbf{x}}j20sin(\beta zcos45°)e^{-j\beta xsin45°} + \hat{\mathbf{z}}20cos(\beta zcos45°)e^{-j\beta xsin45°}$$

$$\mathbf{H}_{\parallel} = -\hat{\mathbf{y}}\frac{\sqrt{10^2 + 10^2}}{377}e^{-j\beta(xsin45° + zcos45°)} - \hat{\mathbf{y}}\frac{\sqrt{10^2 + 10^2}}{377}e^{-j\beta(xsin45° - zcos45°)}$$
$$= -\hat{\mathbf{y}}\frac{\sqrt{200}}{377}\left(e^{-j\beta(xsin45° + zcos45°)} + e^{-j\beta(xsin45° - zcos45°)}\right)$$
$$= -\hat{\mathbf{y}}\frac{\sqrt{200}}{377}\left(e^{j\beta zcos45°} + e^{-j\beta zcos45°}\right)e^{-j\beta xsin45°} = -\hat{\mathbf{y}}2\frac{\sqrt{200}}{377}cos(\beta zcos45°)e^{-j\beta xsin45°}$$

The power density in the parallel polarized wave is:

$$\mathcal{P}_{av}^{\parallel} = Re\left\{\frac{\mathbf{E}_{\parallel}\times\mathbf{H}_{\parallel}^{*}}{2}\right\} =$$

$$Re\left\{\frac{\left(\hat{\mathbf{x}}j20sin(\beta zcos45°)e^{-j\beta xsin45°} + \hat{\mathbf{z}}20cos(\beta zcos45°)e^{-j\beta xsin45°}\right)\times\left(-\hat{\mathbf{y}}2\frac{\sqrt{200}}{377}cos(\beta zcos45°)e^{+j\beta xsin45°}\right)}{2}\right\}$$

$$= \hat{\mathbf{x}}20\frac{\sqrt{200}}{377}cos^2(\beta zcos45°) = \hat{\mathbf{x}}0.75cos^2(\beta zcos45°) \qquad \left[\frac{\text{W}}{\text{m}^2}\right]$$

The total time averaged power density is the sum of the two power densities:

$$\mathcal{P}_{av} = \mathcal{P}_{av}^{\perp} + \mathcal{P}_{av}^{\perp} = \hat{\mathbf{x}}\left[0.53sin^2(\beta zcos45°) + 0.75cos^2(\beta zcos45°)\right] \qquad \left[\frac{\text{W}}{\text{m}^2}\right]$$

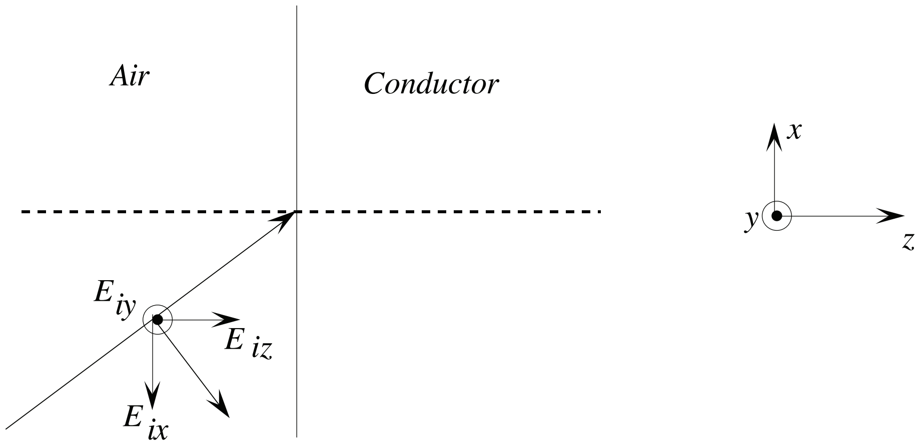


Figure A

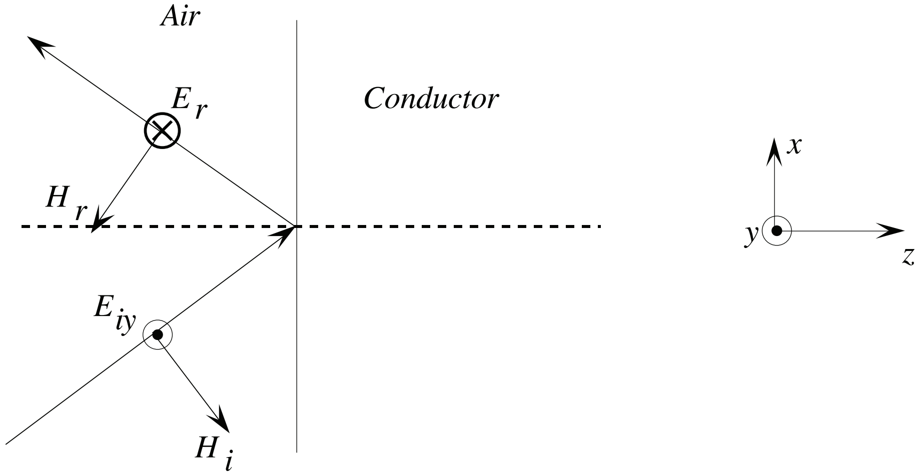


Figure B: The perpendicularly polarized wave

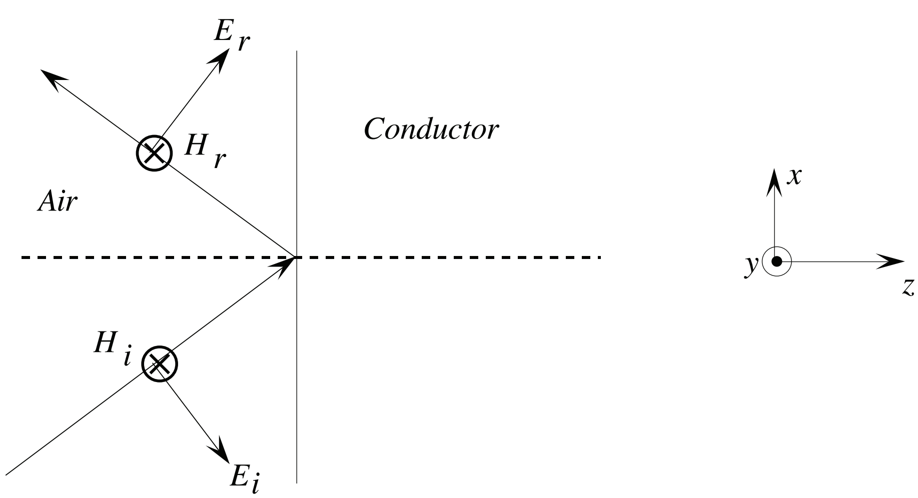


Figure C: The parallel polarized wave

4. An electric field is given as follows:

$$\mathbf{E}(z) = \hat{\mathbf{x}}E_0\left(\cos\beta z\right)e^{-j\beta z} + \hat{\mathbf{y}}jE_0\left(\sin\beta z\right)e^{j\beta z}$$

where β is the phase constant of the wave. Define the type of polarization of the wave as well as its sense of rotation.

Solution:

First, we write the electric field in the time domain:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}E_0(\cos\beta z)\cos(\omega t - \beta z) + \hat{\mathbf{y}}E_0(\sin\beta z)\cos(\omega t + \beta z + \pi/2)$$

Second, we need to decide on a fixed value for z. It is easier to choose a constant value for βz. Usually the choice does not matter but here, because the amplitude is position dependent we must choose a general value, say, $\pi/6$. Note that values like 0 or π /2 would automatically eliminate one component which implies linear polarization. Values like π /4 make the amplitudes of the two terms equal, implying circular polarization. Therefore, for generality, we need to choose a value other than these. With this value, we have:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}E_0(\cos(\pi/6))\cos(\omega t - \pi/6) + \hat{\mathbf{y}}E_0(\sin(\pi/6))\cos(\omega t + \pi/6 + \pi/2)$$

Now we look at how this changes with time. Choosing, arbitrarily, $\omega t = 0$ and $\omega t = \pi/2$ we get:

For $\omega t = 0$:

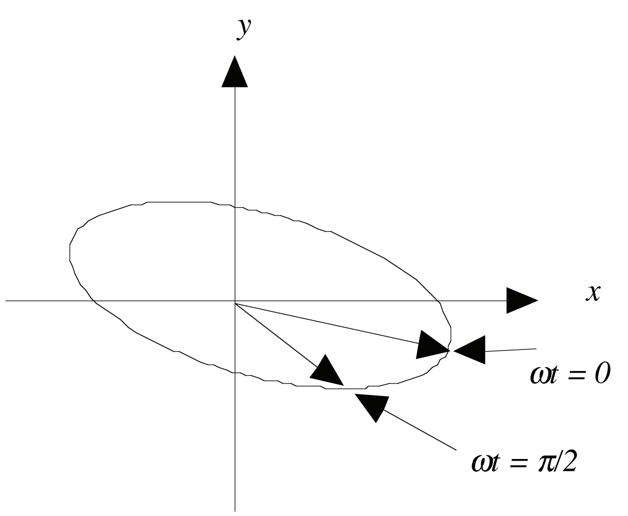
$$\mathbf{E}(z,\omega t=0) = \hat{\mathbf{x}}E_0(\cos(\pi/6))\cos(-\pi/6) + \hat{\mathbf{y}}E_0(\sin(\pi/6))\cos(\pi/6 + \pi/2) = \hat{\mathbf{x}}0.75E_0 - \hat{\mathbf{y}}0.25E_0$$

For $\omega t = 0$:

$$\mathbf{E}(z,\omega t=\pi/2) = \hat{\mathbf{x}}E_0(\cos(\pi/6))\cos(\pi/2 - \pi/6) + \hat{\mathbf{y}}E_0(\sin(\pi/6))\cos(\pi/2 + \pi/6 + \pi/2) = \hat{\mathbf{x}}0.433E_0 - \hat{\mathbf{y}}0.433E_0$$

The figure below shows a sketch of these:

Therefore, this is a left elliptically polarized wave.



Note: at $\beta z = \pi/4$ we would have gotten a left circularly polarized wave which is a special case of elliptical polarization and at 0 or $\pi/2$ a linearly polarized wave (also a special case of elliptical polarization). The polarization of this wave changes as it propagates. Since these are special cases or elliptical polarization it is correct to classify this as elliptically polarized.