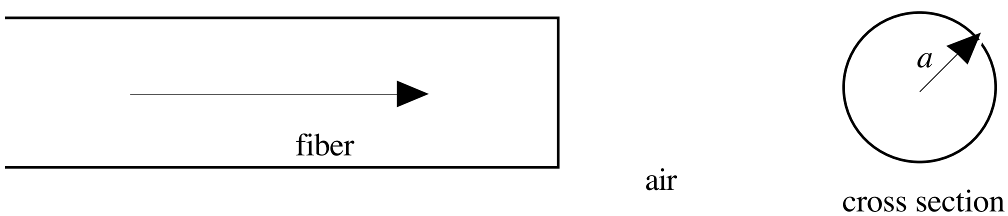
ELECTROMAGNETICS II
March 17, 2004
Second Exam
Solution

Answer the following five questions. Write clearly and concisely. You may use any material, including your notes, book, etc. but you cannot borrow material during the exam. You are encouraged to make <u>reasonable</u> assumptions but you must state them clearly. Only assumptions necessary to solve the problem will be accepted. Permittivity of free space is $\varepsilon_0=8.854\times10^{-12}$ F/m, permeability of free space is $4\pi\times10^{-7}$ H/m.

1. A laser beam propagates in a lossless optical fiber. The fiber is round and has a radius of a=0.1 micrometers. The total time averaged power of the beam is P=0.01[W]. Assume plane wave propagation along the fiber. Now suppose the fiber is cut at some point so that the plane of the cut is clean and straight. If the relative permittivity of the fiber is ε_r=2.25, calculate the power transmitted from the fiber into air. The arrow in the figure on the left shows the direction of propagation of the wave.



Solution: Calculate the power density in the fiber, then the electric field intensity and the transmission coefficient. From these calculate the electric field intensity in air, then the power density and finally, power.

The power density in the fiber is:

$$\mathcal{P}_{av}=\frac{P}{S}=\frac{P}{\pi a^2} \qquad \left[\frac{\text{W}}{\text{m}^2}\right]$$

From this, the electric field intensity in the fiber is:

$$\mathcal{P}_{av}=\frac{E_f^2}{2\eta_f} \qquad \rightarrow \qquad E_f=\sqrt{2\eta_f\mathcal{P}_{av}}=\sqrt{2\eta_f\frac{P}{\pi a^2}} \qquad \left[\frac{\text{V}}{\text{m}}\right]$$

where:

$$\eta_f=\frac{377}{\sqrt{2.25}}=251.334 \qquad [\Omega]$$

The transmission coefficient is

$$T=\frac{2\eta_0}{\eta_0+\eta_f}=\frac{2\times377}{377+377/\sqrt{2.25}}=1.2$$

The electric field intensity in air is:

$$E_a = TE_f \qquad \left[\frac{\text{V}}{\text{m}}\right]$$

Power density in air is:

$$\mathcal{P}_{av} = \frac{E_a^2}{2\eta_a} = \frac{T^2 E_f^2}{2\eta_a} = \frac{T^2 2\eta_f P}{2\eta_a \pi a^2} \qquad \left[\frac{\text{W}}{\text{m}^2}\right]$$

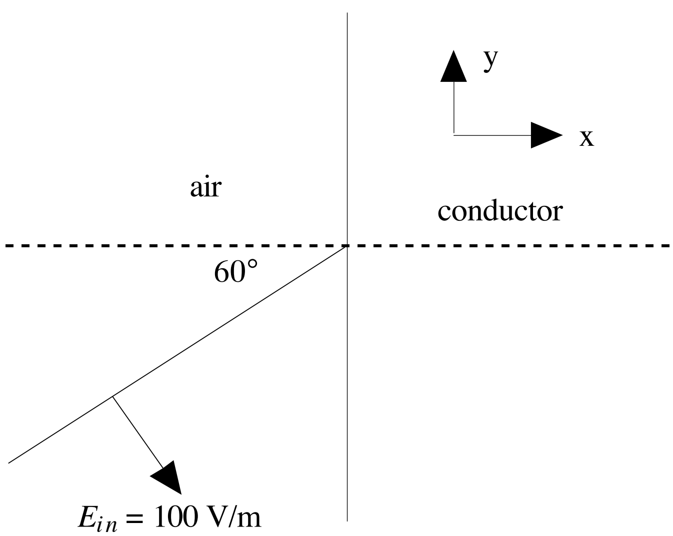
The power in air is this quantity multiplied by the area:

$$P_a = S\mathcal{P}_{av} = \frac{T^2 2\eta_f P \pi a^2}{2\eta_a \pi a^2} = \frac{T^2 \eta_f P}{\eta_a} \qquad [\text{W}]$$

Numerically this gives:

$$P_a = \frac{T^2 \eta_f P}{\eta_a} = \frac{(1.2)^2 \times 251.334 \times 0.1}{377} = 0.0096 \qquad [\text{W}]$$

2. A plane wave propagates in free space (air) and impinges at an angle of 60° on a conducting interface. Calculate the time averaged power density in air. Give both direction and amplitude. Assume the surface of the conductor is at x=0.



Solution: There are a number of ways to solve this problem. The easiest is to write the electric fields and magnetic field by inspection. To do so we write both the incident and reflected fields using **Figure B**. Note that the reflected electric field intensity is such that the tangential components of the incident and reflected field cancel each other on the surface of the conductor as required.

Now we can write each field separately but note that both the incident and reflected magnetic field intensities must be into the page (negative z direction). Note also that the amplitude of the reflected field intensity is the same as that of the incident electric field intensity.

$$E_i = \left[\hat{\mathbf{x}}E_{in}sin60° - \hat{\mathbf{y}}E_{in}cos60°\right]e^{-j\beta(xcos60° + ysin60°)}$$
$$H_i = \hat{\mathbf{z}}\frac{E_{in}}{\eta_0}e^{-j\beta(xcos60° + ysin60°)}$$
$$E_r = \left[\hat{\mathbf{x}}E_{in}sin60° + \hat{\mathbf{y}}E_{in}cos60°\right]e^{j\beta(-xcos60° + ysin60°)}$$
$$H_r = \hat{\mathbf{z}}\frac{E_{in}}{\eta_0}e^{j\beta(-xcos60° + ysin60°)}$$

Now we add the electric fields together and the magnetic field together to find the total field:
For the electric field intensity:

$$E_1 = E_i + E_r = \hat{\mathbf{x}}E_{in}sin60°\left[e^{-j\beta(xcos60° + ysin60°)} + e^{j\beta(-xcos60° + ysin60°)}\right]$$
$$+ \hat{\mathbf{y}}E_{in}cos60°\left[e^{j\beta(-xcos60° + ysin60°)} - e^{-j\beta(xcos60° + ysin60°)}\right]$$
$$= \hat{\mathbf{x}}E_{in}sin60°\left[e^{-j\beta ysin60°} + e^{j\beta ysin60°}\right]e^{-j\beta xcos60°} + \hat{\mathbf{y}}E_{in}cos60°\left[e^{j\beta ysin60°} - e^{-j\beta ysin60°}\right]$$
$$= \hat{\mathbf{x}}E_{in}(sin60°)2cos(\beta ysin60°)e^{-j\beta xcos60°} + \hat{\mathbf{y}}E_{in}(cos60°)j2sin(\beta ysin60°)$$
$$= \hat{\mathbf{x}}2E_{in}(sin60°)cos(\beta ysin60°)e^{-j\beta xcos60°} + \hat{\mathbf{y}}j2E_{in}(cos60°)sin(\beta ysin60°)$$

Similarly for the magnetic field intensity:

$$H_1 = H_i + H_r = \hat{\mathbf{z}}\frac{E_{in}}{\eta_0}\left[e^{-j\beta(xcos60° + ysin60°)} + e^{j\beta(-xcos60° + ysin60°)}\right]$$
$$= \hat{\mathbf{z}}\frac{E_{in}}{\eta_0}\left[e^{-j\beta ysin60°} + e^{j\beta ysin60°}\right]e^{-j\beta xcos60°} = \hat{\mathbf{z}}2\frac{E_{in}}{\eta_0}\left[cos(\beta ysin60°)\right]e^{-j\beta xcos60°}$$

Now we write for the time averaged power density:

$$\mathcal{P}_{av} = Re\left\{\frac{\mathbf{E}\times\mathbf{H}^*}{2}\right\}$$
$$= Re\left\{\frac{\begin{pmatrix}\hat{\mathbf{x}}2E_{in}(sin60°)cos(\beta ysin60°)e^{-j\beta xcos60°}\\ + \hat{\mathbf{y}}j2E_{in}(cos60°)sin(\beta ysin60°)e^{-j\beta xcos60°}\end{pmatrix}\times\left(\hat{\mathbf{z}}2\frac{E_{in}}{\eta_0}\left[cos(\beta ysin60°)\right]e^{j\beta xcos60°}\right)}{2}\right\}$$
$$= Re\left\{\begin{matrix}\hat{\mathbf{x}}\times\hat{\mathbf{z}}\frac{E_{in}}{\eta_0}\left[cos(\beta ysin60°)\right]e^{j\beta xcos60°}2E_{in}(sin60°)cos(\beta ysin60°)e^{-j\beta xcos60°}\\ + \hat{\mathbf{y}}\times\hat{\mathbf{z}}\frac{E_{in}}{\eta_0}\left[cos(\beta ysin60°)\right]e^{j\beta xcos60°}j2E_{in}(cos60°)sin(\beta ysin60°)\end{matrix}\right\}$$
$$= \hat{\mathbf{x}}\times\hat{\mathbf{z}}\frac{E_{in}}{\eta_0}\left[cos(\beta ysin60°)\right]e^{j\beta xcos60°}2E_{in}(sin60°)cos(\beta ysin60°)e^{-j\beta xcos60°}$$
$$= \hat{\mathbf{y}}\frac{2E_{in}^2}{\eta_0}\left[cos^2(\beta ysin60°)\right](sin60°)$$

Note: the above was done in all details. This can be shortened by suing available expressions in the book.

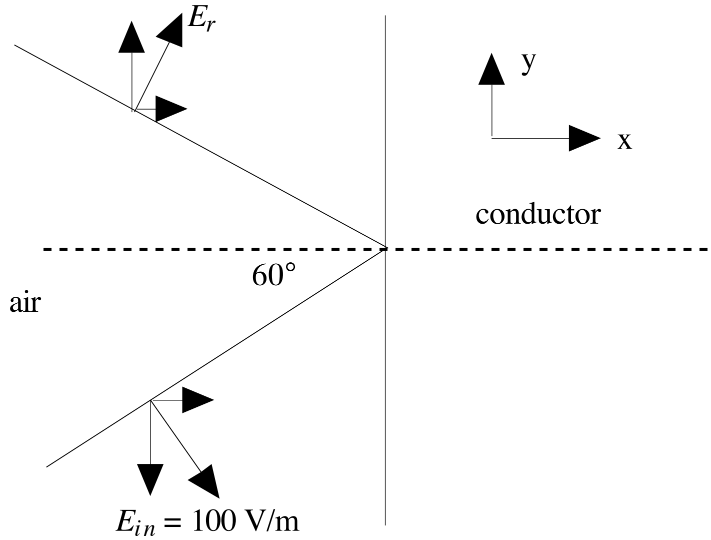


Figure B

3. Given a uniform plane wave:

$$\mathbf{E} = \hat{\mathbf{x}}3e^{-j\beta z} + \hat{\mathbf{y}}4e^{-j\beta z}e^{j\pi/4} \quad V/m$$

Find the polarization of the plane wave as well as its sense of rotation (i.e. left or right). Do not guess. A guess without substantiation will not be accepted.

Solution: follow the standard steps.

(1) Write the electric field intensity in the time domain:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}3cos(\omega t - \beta z) + \hat{\mathbf{y}}4cos(\omega t - \beta z + \pi/4)$$

(2) set z=0:

$$\mathbf{E}(z=0,t) = \hat{\mathbf{x}}3cos(\omega t) + \hat{\mathbf{y}}4cos(\omega t + \pi/4)$$

(3) Choose two distinct values for ωt, say ωt=0 and ωt=π/4. Calculate the electric field for the two values:

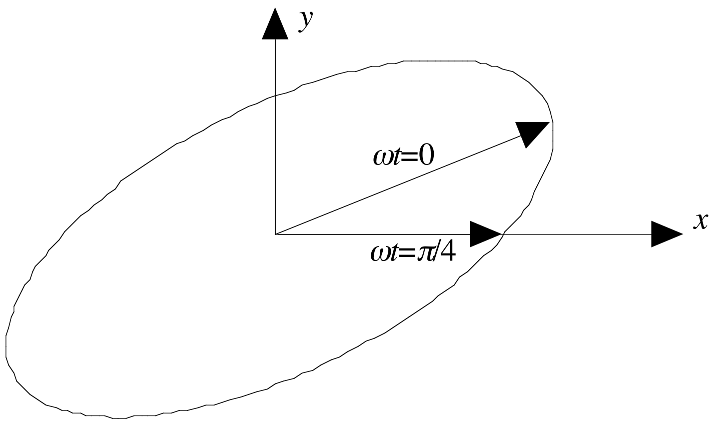
For ωt=0:

$$\mathbf{E}(z=0,\omega t=0) = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}4cos(\pi/4) = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}2.83$$

For ωt=π/4:

$$\mathbf{E}(z=0,\omega t=\pi/4) = \hat{\mathbf{x}}3cos(\pi/4) + \hat{\mathbf{y}}4cos(\pi/4 + \pi/4) = \hat{\mathbf{x}}2.12$$

Thus, the polarization is elliptical since the amplitude changes with rotation. The rotation is clockwise (see Figure). Thus we have a left elliptically polarized wave



4. The frequency of light radiated from the sun is 10^{14} [Hz]. The time averaged power density at the surface of the earth at this frequency is 500 [W/m^2]. Suppose this light impinges on the surface of your skin perpendicularly. The properties of the skin are: conductivity σ=10 [S/m], permeability $\mu=\mu_0$ and permittivity $\varepsilon=2\varepsilon_0$. How far does the light penetrate into your body before its power density is attenuated to 1/1000 of its value in air.

Solution: We first calculate the electric field intensity in air followed by the transmission coefficient from air into the skin. Then, after calculating the power density and the attenuation constant we calculate the depth into the body.
In air:

$$\mathcal{P}_{av}=\frac{E_a^2}{2\eta_0} \quad \rightarrow \quad E_a=\sqrt{2\eta_0\mathcal{P}_{av}} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

The transmission coefficient between air and skin is:

$$T=\frac{2\eta_s}{\eta_0+\eta_s}$$

Now, the intrinsic impedance of skin is

$$\eta_s=\sqrt{\frac{j\omega\mu_s}{\sigma_s+j\omega\varepsilon_s}}=\sqrt{\frac{j2\pi\times10^{14}\times4\pi\times10^{-7}}{10+j2\pi\times10^{14}\times2\times8.854\times10^{-12}}}=\sqrt{\frac{j7.896\times10^{8}}{10+j1.11\times10^{4}}}$$

$$\approx\sqrt{\frac{j7.896\times10^{8}}{j1.11\times10^{4}}}=266.7 \quad [\Omega]$$

This is possible because the conductivity is small in comparison. With these the transmission coefficient is:

$$T=\frac{2\eta_s}{\eta_0+\eta_s}=\frac{2\times266.7}{377+266.7}=0.829$$

The electric field intensity in the skin is:

$$E_s=TE_a \quad \left[\frac{\text{V}}{\text{m}}\right]$$

The power density in the skin at the interface is:

$$\mathcal{P}_s = \frac{E_s^2}{2\eta_s} = \frac{T^2E_a^2}{2\eta_s} = \frac{T^2 2\eta_0 \mathcal{P}_{av}}{2\eta_s} = \frac{T^2\eta_0\mathcal{P}_{av}}{\eta_s} \qquad \left[\frac{\text{W}}{\text{m}^2}\right]$$

This power density is now attenuated with an attenuation constant for a distance d where the power density is 1/1000 of $\mathcal{P}_{av}$. We can write formally:

$$\frac{\mathcal{P}_{av}}{1000} = \frac{T^2\eta_0\mathcal{P}_{av}}{\eta_s} e^{-2\alpha d}$$

Rewriting:

$$\frac{1}{1000} = \frac{T^2\eta_0}{\eta_s} e^{-2\alpha d} \qquad \rightarrow \qquad e^{-2\alpha d} = \frac{\eta_s}{1000T^2\eta_0}$$

Now, taking the natural log on both sides:

$$-2\alpha d = ln\left(\frac{\eta_s}{1000T^2\eta_0}\right) \qquad \rightarrow \qquad d = \frac{1}{-2\alpha} ln\left(\frac{\eta_s}{1000T^2\eta_0}\right)$$

Before we can calculate this we need to attenuation constant. To calculate the constant we need first to decide if we have low or high loss attenuation:

$$\frac{\sigma}{\omega\varepsilon} = \frac{10}{2\pi\times10^{14}\times2\times8.854\times10^{-12}} = 9\times10^{-4}$$

This is a low loss material. The attenuation constant is:

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon_s}} = \frac{10}{2}\sqrt{\frac{4\pi\times10^{-7}}{2\times8.854\times10^{-12}}} = 1331.96 \qquad \left[\frac{\text{Np}}{\text{m}}\right]$$

Now we can write:

$$d = \frac{1}{-2\alpha} ln\left(\frac{\eta_s}{1000T^2\eta_0}\right) = \frac{1}{-2\times1331.96} ln\left(\frac{266.7}{1000\times(0.829)^2\times377}\right) = 2.58\times10^{-3} \qquad [\text{m}]$$

That is, penetration is only 2.58 mm.