ELECTROMAGNETICS II
June 29, 2001
First Exam
(solution)

Answer the following four questions. Write clearly and concisely. You may use any material, including your notes, book, etc. but you cannot borrow material during the exam. You are encouraged to make reasonable assumptions but you must state them clearly. Permittivity of free space is $\varepsilon_0=8.854\times10^{-12}$ F/m, permeability of free space is $4\pi\times10^{-7}$ H/m.

Level of difficulty: Problem 4 is the most difficult, followed by 3, 2 and 1 (easiest).

NOTE: Each problem is worth 20 points. A CORRECT solution to problem 4 will give you an extra 10 points.

1. A conducting wire is made of copper, has radius a and carries a current $I=I_0 sin\omega t$. Assume the current in the conductor is uniformly distributed and the permittivity of the conductor is ε_0. Given: f=1000Hz, I_0=1A, a=2mm and conductivity of copper is $\sigma=5.7\times10^7$ S/m.
Calculate the ratio between the amplitude of the conduction current and the amplitude of the displacement current in the conductor.

Solution: Calculate the current density in the wire (conduction) from which you can calculate the electric field intensity in the conductor. From the electric field intensity, calculate the displacement current density.
The current in the wire is a conduction current. The conduction current density is:

$$J=\frac{I}{s}=\frac{I_0 sin(\omega t)}{\pi a^2} \qquad \left[\frac{\text{A}}{\text{m}^2}\right]$$

where σ is the cross sectional area of the conductor. The electric field intensity in the conductor is:

$$J=\sigma E \qquad \rightarrow \qquad E=\frac{J}{\sigma}=\frac{I_0 sin(\omega t)}{\sigma\pi a^2} \qquad \left[\frac{\text{V}}{\text{m}}\right]$$

The displacement current density is:

$$J_d=\varepsilon_0\frac{dE}{dt}=\frac{\varepsilon_0\omega I_0 cos(\omega t)}{\sigma\pi a^2} \qquad \left[\frac{\text{A}}{\text{m}^2}\right]$$

The total displacement current is the current density multiplied by the cross sectional area of the conductor:

$$I_d=\pi a^2 J_d=\frac{\pi a^2\varepsilon_0\omega I_0 cos(\omega t)}{\sigma\pi a^2}=\frac{\varepsilon_0\omega I_0 cos(\omega t)}{\sigma} \qquad [\text{A}]$$

The ratio between the amplitudes of the conduction and displacement currents is:

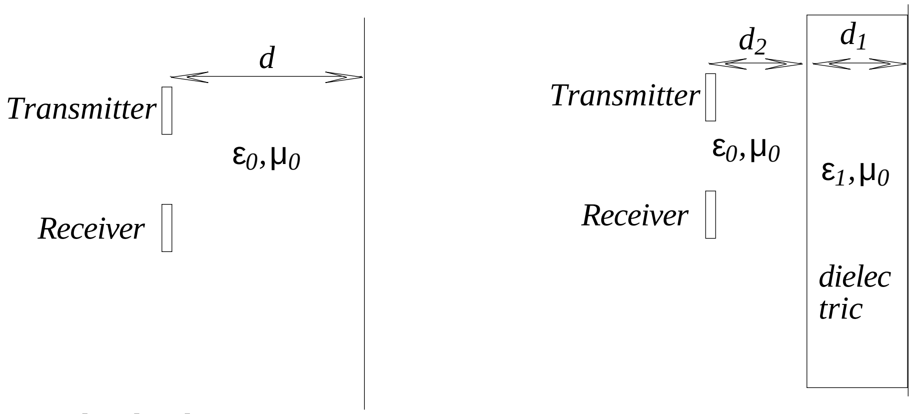
$$\frac{I_c}{I_d}=\frac{I_0}{(\varepsilon_0\omega I_0)/\sigma}=\frac{\sigma}{\varepsilon_0\omega}$$

Numerically this is:

$$\frac{I_c}{I_d}=\frac{\sigma}{\varepsilon_0\omega}=\frac{5.7\times10^7}{8.854\times10^{-12}\times2\times\pi\times1000}=1\times10^{15}$$

That is, the conduction current is 15 orders of magnitude larger than the displacement current (and thus is normally neglected in conducting media)

2. A method of measuring the permittivity of a flat piece of material is as follows: A transmitter is placed *0.3m* from a conducting wall. A very narrow pulse is transmitted and received in the receiver after *2 ns*. Now a flat piece of the dielectric material is placed between the transmitter and the wall and the signal is received after *2.1 ns*. Assume that the only reflections are from the conducting wall and that the dielectric is a perfect dielectric with permeability of free space. Calculate the relative permittivity of the dielectric.



Note: $d_1+d_2=d=0.3m$.

Solution: The increase in time of propagation to and from the wall is due to the larger permittivity of the dielectric.

Before the dielectric has been inserted, the phase velocity is:

$$v_{p0} = \frac{2d}{t} = \frac{0.6}{2\times10^{-9}} = 3\times10^{8} \quad \left[\frac{m}{s}\right]$$

After inserting the dielectric:

$$t = \frac{2d_2}{v_{p0}} + \frac{2d_1}{v_{p1}} \qquad \rightarrow \qquad \frac{2d_1}{v_{p1}} = t - \frac{2d_2}{v_{p0}}$$

or:

$$\frac{2d_1\sqrt{\varepsilon_r\varepsilon_0\mu_0}}{1} = \frac{2d_1\sqrt{\varepsilon_r}}{c} = t - \frac{2d_2}{v_{p0}}$$

Thus:

$$\sqrt{\varepsilon_r} = \frac{ct}{2d_1} - \frac{2d_2c}{v_{p0}2d_1} = \frac{ct}{2d_1} - \frac{d_2}{d_1}$$

Example: if $d_2=d_1=0.15$m:

$$\sqrt{\varepsilon_r} = \frac{3\times10^{8}\times2.1\times10^{-9}}{2\times0.15} - 1 = 1.1 \qquad \rightarrow \qquad \varepsilon_r = 1.21$$

3. A high voltage power line is made of aluminum, has a radius of 20mm and carries a sinusoidal current of 1000 A at 60 Hz. This current is a conduction current. The conductivity of aluminum is 3.6×10^{7} S/m. Assuming the permittivity of aluminum is the same as that of free space, calculate the displacement current in the power line.

Solution: Calculate the conduction current density in the conductor. From this calculate the electric field intensity. From the electric field intensity calculate the displacement current density and then the displacement current.

The conduction current density is the total conduction current divided by the cross sectional area of the conductor:

$$J_c = \frac{I}{S} = \frac{I sin\omega t}{\pi r^2} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right]$$

The electric field intensity in the conductor is:

$$J_c = \sigma E \qquad \rightarrow \qquad E = \frac{J_c}{\sigma} = \frac{I\, sin\omega t}{\sigma \pi r^2} \qquad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

Now we can calculate the displacement current density as:

$$J_d = \varepsilon_0 \frac{dE}{dt} = \varepsilon_0 \frac{d}{dt}\left[\frac{I\, sin\omega t}{\sigma \pi r^2}\right] = \frac{I \varepsilon_0 \omega}{\sigma \pi r^2} cos\ \omega t \qquad \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right]$$

The displacement current is the displacement current density multiplied by cross sectional area:

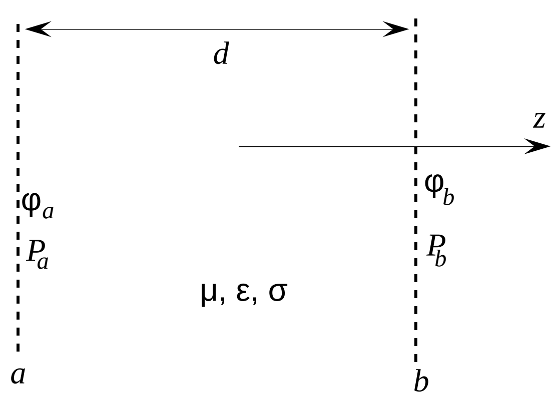
$$I_d = J_d S = \frac{I \varepsilon_0 \omega \pi r^2}{\sigma \pi r^2} cos\ \omega t = \frac{I \varepsilon_0 \omega}{\sigma} cos\ \omega t \qquad [\mathrm{A}]$$

With $\omega = 2\pi \times 60 = 120\pi$, $\varepsilon_0 = 8.854\times 10^{-12}$, I=1000 and $\sigma = 3.6\times 10^7$ we get:

$$I_d = \frac{I \varepsilon_0 \omega}{\sigma} cos\ \omega t = \frac{1000 \times 8.854\times 10^{-12} \times 120\pi}{3.6\times 10^7} cos\ 120\pi t = 9.27\times 10^{-14} \qquad [\mathrm{A}]$$

Note: the very small displacement current compared with the conduction current is the reason we normally do not consider displacement currents in conductors.

4. A low loss dielectric material is known to have permeability of free space but its conductivity and permittivity are not knows. Two measurements are performed to determine the permittivity and conductivity of the material
1. A plane wave is generated so it propagates in the direction indicated in the figure and the time averaged power density is measured at point (a) and (b). The values obtained are P_a and P_b.
2. The phase of the plane wave is measured at points (a) and (b). The values obtained are φ_a and φ_b.
Calculate the permittivity and conductivity of the material if the measurements are performed at a frequency ω.



Solution: We can write the attenuation and phase constants from the fact that the material is low loss, assuming unknown conductivity s and permittivity e. From the power densities at the two locations we can write in general:

At any point in the material we can write the following:

$$P(z) = Pe^{-2\alpha z}e^{-j2\beta z}$$

At $z=a$ we have:

$$P(z=a) = Pe^{-2\alpha a}e^{-j2\beta a}$$

At $z=b$ we have:

$$P(z=a) = Pe^{-2\alpha b}e^{-j2\beta b}$$

Now we know that:

$$Pe^{-2\alpha a} = P_a \qquad and \qquad Pe^{-2\alpha b} = P_b$$

Since P_a and P_b are given we write:

$$Pe^{-2\alpha a} = P_a \qquad and \qquad \frac{P_b}{P_a} = \frac{Pe^{-2\alpha b}}{Pe^{-2\alpha a}} = \frac{e^{-2\alpha b}}{e^{-2\alpha a}} = e^{-2\alpha b}e^{2\alpha a} = e^{-2\alpha(b-a)}$$

or:

$$\frac{P_b}{P_a} = e^{-2\alpha(b-a)} \qquad \rightarrow \qquad ln\left(\frac{P_b}{P_a}\right) = -2\alpha(b-a) = -2\alpha d$$

From this we write:

$$\alpha = -\frac{1}{2d}\, ln\left(\frac{P_b}{P_a}\right) = \frac{1}{2d}\, ln\left(\frac{P_a}{P_b}\right)$$

The attenuation constant for low loss dielectrics is also equal to:

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon}}$$

Thus the first relation we have is:

$$\boxed{\frac{1}{d}\, ln\left(\frac{P_a}{P_b}\right) = \sigma\sqrt{\frac{\mu_0}{\varepsilon}}}$$

In this relation the unknowns are s and e. We need another relation and this is found from the phase. We write:

$$\varphi_a = \beta a, \qquad \varphi_b = \beta b$$

The change in phase may be written as:

$$\varphi_b - \varphi_a = \beta b - \beta a = \beta(b-a) = \beta d$$

or:

$$\varphi_b - \varphi_a = \beta d$$

For a low loss dielectric we have:

$$\beta = \omega\sqrt{\mu_0\varepsilon}$$

Thus:

$$\boxed{\varphi_b - \varphi_a = \omega d \sqrt{\mu_0 \varepsilon}}$$

In this relation the only unknown is e so we can write:

$$\varepsilon = \frac{(\varphi_b - \varphi_a)^2}{\omega^2 d^2 \mu_0}$$

Substituting this into the first relation, we write:

$$\sigma = \frac{1}{d} \sqrt{\frac{\varepsilon}{\mu_0}}\, ln\left(\frac{P_a}{P_b}\right) = \frac{1}{d} \sqrt{\frac{(\varphi_b - \varphi_a)^2}{\omega^2 d^2 \mu_0^2}}\, ln\left(\frac{P_a}{P_b}\right) = \frac{1}{d} \frac{(\varphi_b - \varphi_a)}{\omega d \mu_0}\, ln\left(\frac{P_a}{P_b}\right)$$

Thus, the conductivity and permittivity of the material are:

$$\sigma = \frac{1}{d} \frac{(\varphi_b - \varphi_a)}{\omega d \mu_0}\, ln\left(\frac{P_a}{P_b}\right) \qquad \text{and} \qquad \varepsilon = \frac{(\varphi_b - \varphi_a)^2}{\omega^2 d^2 \mu_0}$$
