A magnetic field intensity is defined in terms of a vector potential as follows:

\[ \mathbf{H} = -j\omega \mathbf{F} \]

where \( \mathbf{F} \) is a vector potential. Assume a source free environment.

**a.** What would you call this vector potential? Explain your reasons.

**Start with Ampere’s law:**

\[ \nabla \times \mathbf{H} = j\omega \mathbf{D} \]

Substituting the given magnetic field intensity into Ampere’s law:

\[ \nabla \times \mathbf{H} = \nabla \times (-j\omega \mathbf{F}) = j\omega \nabla \times (-\mathbf{F}) = j\omega \mathbf{D} \]

Thus:

\[ \mathbf{D} = -\nabla \times \mathbf{F} \]

Since the curl of \( \mathbf{F} \) defines the electric flux density, \( \mathbf{F} \) is an electric vector potential.

**b.** Write Maxwell’s equations in terms of this vector potential alone. What is the necessary gauge to be able to write Maxwell’s equations as a Helmholtz equation in terms of \( \mathbf{F} \)?

**Solution:**

**a.** The vector potential \( \mathbf{F} \) is an electric vector potential for the following reason.

Start with Faraday’s law:

\[ \nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} = -j\omega \mu (-j\omega \mathbf{F}) = -\omega^2 \mu \mathbf{F} \]

From Ampere’s law:

\[ \nabla \times \mathbf{H} = j\omega \varepsilon \mathbf{E} \]

Taking the curl on both sides of this equation:

\[ \nabla \times (\nabla \times \mathbf{H}) = j\omega \varepsilon (\nabla \times \mathbf{E}) \]

Substituting for \( \nabla \times \mathbf{E} \) from Faraday’s law:
\[ \nabla \times (\nabla \times H) = j \omega \epsilon (-\omega^2 \mu F) \]

Substituting for \( H \) from the given field:

\[ \nabla \times (\nabla \times (-j \omega F)) = j \omega \epsilon (-\omega^2 \mu F) \]

or:

\[ -j \omega \nabla \times (\nabla \times F) = -j \omega \epsilon (\omega^2 \mu F) \]

or:

\[ \nabla \times (\nabla \times F) = \omega^2 \mu \epsilon F \]

Expanding the left hand side:

\[ -\nabla^2 F + \nabla (\nabla \cdot F) - \omega^2 \mu \epsilon F = 0 \]

To obtain a Helmholtz equation, we must assume \( \nabla \cdot F = 0 \) (this is the gauge in this case). Thus, the Helmholtz equation is:

\[ \nabla^2 F + \omega^2 \mu \epsilon F = 0 \]

2. A plane wave is given, with the following electric field intensity:

\[ E = \hat{x} 100 e^{-j2\pi \times 10^9 y} + \hat{z} 200 e^{-j2\pi \times 10^9 y} \text{ V/m} \]

a. Find the magnetic field intensity of the wave.

b. Find the time averaged power density propagated by the wave.

**Solution:** Write the magnetic field intensity from the fact that \( H \) must be perpendicular to \( E \), both \( H \) and \( E \) must be perpendicular to the direction of propagation and the wave propagates in the positive \( y \) direction.

a. \( H \) must propagate in the positive \( y \) direction, its magnitude is \( E / \eta_0 \) and its components are as follows:

1. The \( x \) component of \( E \) produces a negative \( z \) component of \( H \) (\( \hat{x} \times \hat{z} = \hat{y} \)).
2. The \( z \) component of \( E \) produces a positive \( x \) component of \( H \) (\( \hat{z} \times \hat{x} = \hat{y} \)). Thus, with \( \eta_0 = 377 \Omega \), we can write directly:

\[ H = -\hat{z} \frac{100}{377} e^{-j2\pi \times 10^9 y} + \hat{x} \frac{200}{377} e^{-j2\pi \times 10^9 y} \text{ A/m} \]

**Note:** you can obtain exactly the same solution by substituting \( E \) into Faraday’s law but this is lengthier.

b. From the Poynting vector:
\[ P_{av} = \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H}^* \} \]

\[ = \frac{1}{2} \text{Re} \{ [x 100 e^{-j2\pi \times 10^9 y} + z 200 e^{-j2\pi \times 10^9 y}] \left\{ -\frac{x 100}{377} e^{j2\pi \times 10^9 y} + \frac{z 200}{377} e^{j2\pi \times 10^9 y} \right\} \} \]

\[ = \frac{1}{2} \text{Re} \left\{ \frac{x 100^2}{377} + \frac{z 200^2}{377} \right\} = \frac{5 \times 100^2}{2 \times 377} = 66.3 \quad \text{[W/m}^2]\]

3. A capacitor C is connected to a dc source through a conducting wire which has conductivity \( \sigma \), radius \( a \) and a total resistance \( R \). The switch is connected at time \( t=0 \).

Calculate:

a. The displacement current density in the capacitor as a function of time. Plot.

b. The displacement current density in the conductor as a function of time. Plot.

![Diagram of a circuit with a capacitor and a resistor](image)

a. As a circuit, the current density in the wires of the circuit is:

\[ J(t) = \frac{I(t)}{S} = \frac{V}{\pi a^2 R} e^{-t/RC} \]

The total displacement current in the capacitor is:

\[ I_c = I(t) = \frac{V}{R} e^{-t/RC} \]

The displacement current density is this current divided by the area of the capacitor, \( A \):

\[ J_d = J_c = \frac{V}{R A} e^{-t/RC} \]

![Graph of current density \( J_d \) as a function of time](image)
b. In the conductor:

\[ J_d = \varepsilon \frac{\partial E}{\partial t} \]

where \( J(t) = \sigma E(t) \) or:

\[ E(t) = \frac{J(t)}{\sigma} = \frac{V}{\sigma \pi a^2 R} \quad e^{-t/RC} \]

In the conductor, \( \varepsilon = \varepsilon_0 \):

\[ J_d = \varepsilon \frac{\partial E}{\partial t} = -\frac{V \varepsilon}{\sigma \pi a^2 R^2 C} \quad e^{-t/RC} \]

![Current density in conductor \( J_d \) as a function of time.](image)

4. Two parallel plate conductors are given, each \( w = 0.1 \text{m} \) wide and separated a distance of \( d = 0.01 \text{m} \). The magnetic field intensity is horizontal and uniform everywhere between the plates and the electric field intensity is vertical and uniform everywhere between the plates. Outside the plates (outside the rectangle of dimensions \( w \) by \( d \)) the electric field intensity and the magnetic field intensity are zero.

This structure propagates 100 W (time averaged power) in what is essentially a plane wave. Calculate the current (magnitude and direction) in the upper and lower conductors that supply this power.

![Diagram of two parallel plate conductors](image)
**Solution:** Start with the definition of power and power density from the Poynting theorem.

In a plane wave we can write:

\[ P_{av} = \frac{E^2}{2\eta} = \frac{\eta H^2}{2} \quad \text{[W/m}^2\text{]} \]

where \( E = \eta H \) was used to obtain the second form. Now, since \( E \) and \( H \) are uniform, the power density is also uniform (independent of position within the rectangle). The total power is the power density multiplied by area:

\[ P = P_{av} \times S = \frac{\eta H^2 wd}{2} \quad \text{[W]} \]

Since \( P \) is known as is everything else except \( H \), we get:

\[ H = \sqrt{\frac{2P}{\eta wd}} \quad \text{[A/m]} \]

Now, from Ampere’s law, taking a contour around the upper conductor (see Figure below):

\[ I = \oint \mathbf{H} \cdot d\mathbf{l} = \sqrt{\frac{2P}{\eta wd}} \times w \quad \text{[A]} \]

For the given values:

\[ I = \sqrt{\frac{2 \times 100}{377 \times 0.1 \times 0.01}} \times 0.1 = 0.728 \quad \text{[A]} \]