Electromagnetics II
Exam 1
February 23, 2000

Solve the following four problems. Each carries equal weight. Be concise and clear. If you need to assume anything, feel free to do so but only reasonable assumptions that do not alter the problem will be accepted. All materials are free space unless indicated otherwise.

1. A magnetic field intensity is defined in terms of a vector potential as follows:

$$\mathbf{H} = -j\omega \mathbf{F}$$

where **F** is a vector potential. Assume a source free environment.
a. What would you call this vector potential? Explain your reasons.
b. Write Maxwell's equations in terms of this vector potential alone. What is the necessary gauge to be able to write Maxwell's equations as a Helmholtz equation in terms of **F**?

Solution:
a. The vector potential **F** is a an electric vector potential for the following reason.
Start with Ampere's law:

$$\times \mathbf{H} = j\omega \mathbf{D}$$

Substituting the given magnetic field intensity into Ampere's law:

$$\times \mathbf{H} = \quad \times(-j\omega \mathbf{F}) = j\omega \quad \times(-\mathbf{F}) = j\omega \mathbf{D}$$

Thus:

$$\mathbf{D} = - \quad \times \mathbf{F}$$

Since the curl of **F** defines the electric flux density, **F** is an electric vector potential.

b. Start with Faraday's law:

$$\times \mathbf{E} = -j\omega\mu \mathbf{H} = -j\omega\mu(-j\omega \mathbf{F}) = -\omega^2 \mu \mathbf{F}$$

From Ampere's law:

$$\times \mathbf{H} = j\omega\varepsilon \mathbf{E}$$

Taking the curl on both sides of this equation:

$$\times(\quad \times \mathbf{H}) = j\omega\varepsilon(\quad \times \mathbf{E})$$

Substituting for $\quad \times \mathbf{E}$ from Faraday's law:

$$\nabla\times(\nabla\times\mathbf{H})=j\omega\varepsilon(-\omega^2\mu\mathbf{F})$$

Substituting for **H** from the given field:

$$\nabla\times(\nabla\times(-j\omega\mathbf{F})=j\omega\varepsilon(-\omega^2\mu\mathbf{F})$$

or:

$$-j\omega\nabla\times(\nabla\times\mathbf{F})=-j\omega\varepsilon(\omega^2\mu\mathbf{F})$$

or:

$$\nabla\times(\nabla\times\mathbf{F})=\omega^2\mu\varepsilon\mathbf{F}$$

Expanding the left hand side:

$$-\nabla^2\mathbf{F}+\nabla(\nabla.\mathbf{F})-\omega^2\mu\varepsilon\mathbf{F}=0$$

To obtain a Helmholtz equation, we must assume $\nabla.\mathbf{F}=0$ (this is the gauge in this case). Thus, the Helmholtz equation is:

$$\nabla^2\mathbf{F}+\omega^2\mu\varepsilon\mathbf{F}=0$$

2. A plane wave is given, with the following electric field intensity:

$$\mathbf{E}=\hat{\mathbf{x}}100e^{-j2\pi\times10^9 y}+\hat{\mathbf{z}}200e^{-j2\pi\times10^9 y}\qquad\left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

a. Find the magnetic field intensity of the wave.
b. Find the time averaged power density propagated by the wave.

Solution: Write the magnetic field intensity from the fact that **H** must be perpendicular to **E**, both **H** and **E** must be perpendicular to the direction of propagation and the wave propagates in the positive *y* direction.

a. H must propagate in the positive *y* direction, its magnitude is E/η_0 and its components are as follows:

1. The *x* component of **E** produces a negative *z* component of **H** ($\hat{\mathbf{x}}\times(-\hat{\mathbf{z}})=\hat{\mathbf{y}}$).

2. The *z* component of **E** produces a positive *x* component of **H** ($\hat{\mathbf{z}}\times\hat{\mathbf{x}}=\hat{\mathbf{y}}$). Thus, with $\eta_0=377\ \Omega$, we can write directly:

$$\mathbf{H}=-\hat{\mathbf{z}}\frac{100}{377}e^{-j2\pi\times10^9 y}+\hat{\mathbf{x}}\frac{200}{377}e^{-j2\pi\times10^9 y}\qquad\left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

Note: you can obtain exactly the same solution by substituting **E** into Faraday's law but this is lengthier.

b. From the Poynting vector:

$$\mathbf{P}_{av}=\frac{1}{2}Re\left\{\mathbf{E}\times\mathbf{H}^*\right\}$$

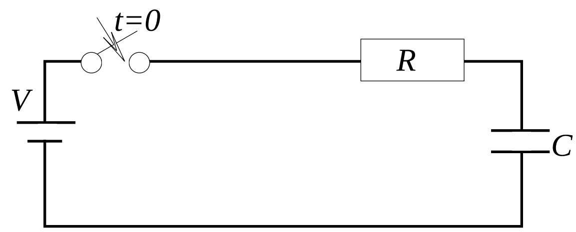
$$=\frac{1}{2}Re\left\{\left(\hat{\mathbf{x}}100e^{-j2\pi\times10^9y}+\hat{\mathbf{z}}200e^{-j2\pi\times10^9y}\right)\times\left(-\hat{\mathbf{z}}\frac{100}{377}e^{j2\pi\times10^9y}+\hat{\mathbf{x}}\frac{200}{377}e^{j2\pi\times10^9y}\right)\right\}$$

$$=\frac{1}{2}Re\left\{\hat{\mathbf{y}}\frac{100^2}{377}+\hat{\mathbf{y}}\frac{200^2}{377}\right\}=\hat{\mathbf{y}}\frac{5\times100^2}{2\times377}=\hat{\mathbf{y}}66.3 \qquad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

3. A capacitor C is connected to a dc source through a conducting wire which has conductivity σ, radius a and a total resistance R. The switch is connected at time t=0. Calculate:

a. The displacement current density in the capacitor as a function of time. Plot.

b. The displacement current density in the conductor as a function of time. Plot.



a. As a circuit, the current density in the wires of the circuit is:

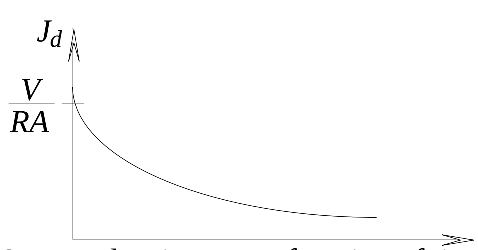
$$J(t)=\frac{I(t)}{S}=\frac{V}{\pi a^2R}e^{-t/RC}$$

The total displacement current in the capacitor is:

$$I_c=I(t)=\frac{V}{R}e^{-t/RC}$$

The displacement current density is this current divided by the area of the capacitor, A:

$$J_d=J_c=\frac{V}{RA}e^{-t/RC}$$



Current density J_d as a function of time.

b. In the conductor:

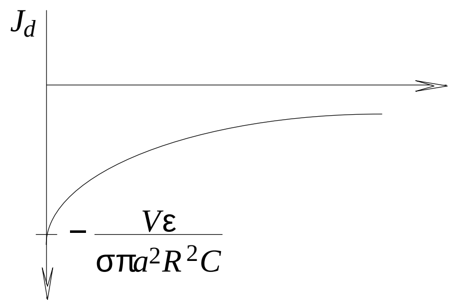
$$\mathbf{J}_d = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

where J(t)=σE(t) or:

$$E(t) = \frac{J(t)}{\sigma} = \frac{V}{\sigma \pi a^2 R} e^{-t/RC}$$

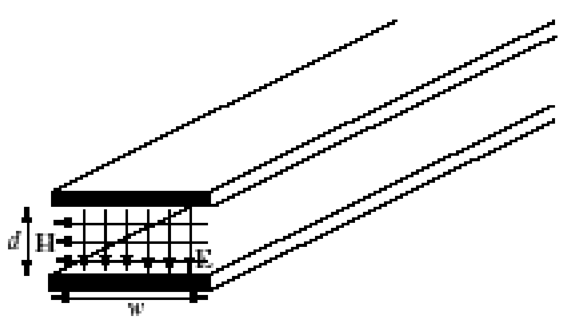
In the conductor, $\varepsilon=\varepsilon_0$:

$$J_d = \varepsilon \frac{\partial E}{\partial t} = -\frac{V\varepsilon}{\sigma \pi a^2 R^2 C} e^{-t/RC}$$



Current density in conductor (J_d) as a function of time.

4. Two parallel plate conductors are given, each w=0.1m wide and separated a distance of d=0.01m. The magnetic field intensity is horizontal and uniform everywhere between the plates and the electric field intensity is vertical and uniform everywhere between the plates. Outside the plates (outside the rectangle of dimensions w by d) the electric field intensity and the magnetic field intensity are zero.
This structure propagates 100 W (time averaged power) in what is essentially a plane wave. Calculate the current (magnitude and direction) in the upper and lower conductors that supply this power.



Solution: Start with the definition of power and power density from the Poynting theorem.

In a plane wave we can write:

$$P_{av} = \frac{E^2}{2\eta} = \frac{\eta H^2}{2} \qquad \left[\frac{W}{m^2}\right]$$

where $E = \eta H$ was used to obtain the second form. Now, since E and H are uniform, the power density is also uniform (independent of position within the rectangle). The total power is the power density multiplied by area:

$$P = P_{av} \times S = \frac{\eta H^2 wd}{2} \qquad [\mathrm{W}]$$

Since P is known as is everything else except H, we get:

$$H = \sqrt{\frac{2P}{\eta wd}} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

Now, from Ampere's law, taking a contour around the upper conductor (see Figure below):

$$I = \oint \mathbf{H} . d\mathbf{l} = \sqrt{\frac{2P}{\eta wd}} \times w \qquad [\mathrm{A}]$$

For the given values:

$$I = \sqrt{\frac{2P}{\eta wd}} \times w = \sqrt{\frac{2 \times 100}{377 \times 0.1 \times 0.01}} \times 0.1 = 0.728 \qquad [\mathrm{A}]$$