Summary Chapter 12.

The fundamentals of wave propagation and the behavior of waves in various media are the subjects of the present chapter. We start with the source free electromagnetic wave equation in general, lossy and lossless media (see **Examples 12.13, 12.14** and **Exercises 12.3** and **12.4**):

$$\nabla^2\mathbf{E} = j\omega\mu(\sigma\mathbf{E} + j\omega\varepsilon\mathbf{E}) \quad (12.16) \qquad \text{and:} \qquad \nabla^2\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = 0 \quad (12.18)$$

Wave equations identical in form may be written for **H**, **B**, **D**, **A** or V and may also be written in the time domain.

Uniform plane waves: are waves in which the amplitude and phase are constant at any point on the plane perpendicular to the direction of propagation of the wave. The form we assume is $\mathbf{E} = \hat{\mathbf{x}}E(z)$.

Solution of the lossless wave equation (**Eq. (12.18)**) for plane waves in lossless media is:

$$E_x(z) = E_0^+ e^{-jkz} + E_0^- e^{jkz} \quad [\text{V/m}] (12.24), \qquad k = \omega\sqrt{\mu\varepsilon} \quad [\text{rad/m}] \quad (12.23)$$

or, in the time domain:

$$E_x(z,t) = \text{Re}\left\{E_x(z)e^{j\omega t}\right\} E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz) \quad [\text{V/m}] \quad (12.25)$$

An arbitrary phase angle ϕ may also be added to either solution. The first term is a forward propagating wave (in the z-direction), the second a backward propagating wave (negative z-direction in this case).

Properties of the wave:

$$\text{Phase velocity}: \ v_p = \frac{1}{\sqrt{\mu\varepsilon}} \quad \left[\frac{\text{m}}{\text{s}}\right] \quad (12.28) \qquad \text{In free space:} \quad v_p \approx 3\times10^8 \quad \left[\frac{\text{m}}{\text{s}}\right] \quad (12.29)$$

$$\text{Wavelength:} \quad \lambda = \frac{v_p}{f} = \quad [\text{m}] \quad (12.30) \qquad \text{Wavenumber:} \quad k = \frac{2\pi}{\lambda} \quad \left[\frac{\text{rad}}{\text{m}}\right] \quad (12.31)$$

$$\text{Intrinsic impedance:} \ \eta = \sqrt{\frac{\mu}{\varepsilon}} \quad [\Omega] \quad (12.37) \qquad \text{In free space:} \ \eta_0 \approx 377 \quad [\Omega] \quad (12.39)$$

Poynting theorem, Poynting vector, power and power density

The **Poynting vector** gives the magnitude and direction of propagation of the instantaneous power density:

$$\boldsymbol{\mathcal{P}} = \mathbf{E}\times\mathbf{H} \quad [\text{W/m}^2] \quad (12.52)$$

The **Poynting theorem** gives the net power entering or leaving a volume v, enclosed by area s:

$$\mathcal{P}(t) = \oint_s (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_v \left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right) dv - \int_v \mathbf{E}\cdot\mathbf{J}\, dv \quad [\text{W}] \quad (12.51)$$

The first term on the right hand side is the time rate of change of energy, the second is power due to source and induced currents. A net negative power (power flow into the volume) is the receiver case. Net positive power (out of the volume) is the transmitter case.

Time averaged power density can be calculated from instantaneous power density or from the complex Poynting vector:

$$\mathcal{P}_{av} = \frac{1}{T}\int_0^T \mathcal{P}(t)\,dt \quad \left[\frac{\text{W}}{\text{m}^2}\right] \quad (12.55) \quad \text{or:} \quad \mathcal{P}_{av} = \frac{1}{2}\text{Re}\{\mathbf{E}\times\mathbf{H}^*\} \quad \left[\frac{\text{W}}{\text{m}^2}\right] \quad (12.58)$$

where $T = 1/f = 2\pi/\omega$. The complex Poynting vector is $\mathcal{P}_c = \mathbf{E}\times\mathbf{H}^*$ [W/m^2]. The complex Poynting theorem may be written as

$$\oint_s (\mathbf{E}\times\mathbf{H}^*)\cdot d\mathbf{s} = j\omega\int_v (\varepsilon\mathbf{E}\cdot\mathbf{E}^* - \mu\mathbf{H}\cdot\mathbf{H}^*)dv - \int_v \mathbf{E}\cdot\mathbf{J}^*\,dv \quad [\text{W}] \quad (12.69)$$

where $\mathbf{E}\cdot\mathbf{J}^*$ may be negative or positive depending on the source of **J***. Time averaged power is usually calculated using **Eq. (12.69).**
Time averaged energy densities (electric and magnetic) are

$$w_{e(av)} = \frac{\varepsilon\mathbf{E}\cdot\mathbf{E}^*}{4} \qquad w_{m(av)} = \frac{\mu\mathbf{H}\cdot\mathbf{H}^*}{4} \qquad (12.74)$$

Propagation of plane waves in general media: Given properties (ε,μ,σ) the wave equation is written in terns of the complex permittivity ε_c as:

$$\nabla^2\mathbf{E} = j\omega\mu(j\omega\varepsilon_c)\mathbf{E} \quad (12.81), \qquad \text{where} \quad \varepsilon_c = \varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right] \quad (12.79)$$

The wave equation to solve is:

$$\nabla^2\mathbf{E} - \gamma^2\mathbf{E} = 0 \quad (12.84), \quad \text{where} \quad \gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} \quad (12.83)$$

and:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} \quad \left[\frac{\text{Np}}{\text{m}}\right] \quad (12.95), \qquad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]} \quad \left[\frac{\text{rad}}{\text{m}}\right] \quad (12.96)$$

$\gamma = \alpha + j\beta$ is the **propagation constant**, α is the **attenuation constant** and β the **phase constant**. Phase velocity and wavelength are also dependent on conductivity see **Eqs. (12.97)** and **(12.98)**).
The **intrinsic impedance** is now complex:

$$\eta = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \quad [\Omega] \quad (12.102)$$

The main effect which is different than propagation in lossless media is attenuation of the waves.
Solution for attenuated plane waves:

$$E_x(z) = E_0^+ e^{-\alpha z}e^{-j\beta z} + E_0^- e^{\alpha z}e^{j\beta z} \quad [\text{V/m}] \quad (12.92)$$

or, in the time domain:

$$E_x(z,t) = E_0^+ e^{-\alpha z}\cos(\omega t - \beta z) + E_0^- e^{\alpha z}\cos(\omega t + \beta z) \quad [\text{V/m}] \quad (12.93)$$

Low loss dielectrics: $\sigma/\omega\varepsilon << 1$. Approximations:

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \quad \left[\frac{\text{Np}}{\text{m}}\right] \quad (12.104) \qquad \beta \approx \omega\sqrt{\mu\varepsilon}\left(1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right) \quad \left[\frac{\text{rad}}{\text{m}}\right] (12.105)$$

$$\eta \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1+\frac{j\sigma}{2\omega\varepsilon}\right) \quad [\Omega] \quad (12.109)$$

High loss materials: $\sigma / \omega\varepsilon >> 1$. Approximations:

$$\alpha \approx \sqrt{\pi f \mu \sigma} \quad \left[\frac{\text{Np}}{\text{m}}\right], \quad \beta \approx \sqrt{\pi f \mu \sigma} \quad \left[\frac{\text{rad}}{\text{m}}\right] \quad (12.111)$$

Skin depth is the depth at which the amplitude of the wave reduces to $1/e$ of its value at the surface:

$$\delta \approx \sqrt{\frac{1}{\pi f \mu \sigma}} = \frac{1}{\alpha} \quad [\text{m}] \quad (12.113)$$

Intrinsic impedance:

$$\eta \approx (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\frac{1}{\sigma\delta} \quad [\Omega] \quad (12.116)$$

Group velocity is the velocity of a packet of waves in a narrow range of frequencies. It is different from phase velocity except in perfect dielectrics

$$v_g = \frac{1}{d\beta / d\omega} \quad \left[\frac{\text{m}}{\text{s}}\right] \quad (12.122)$$

Dispersion is the frequency dependence of the phase velocity which causes waves of different frequencies to travel at different velocities. Perfect dielectrics are dispersionless.

Polarization of plane waves is the path described by the tip of the electric field intensity as it propagates in space towards the observer

1. Linear polarization – the tip of the electric field intensity describes a line
2. Circular polarization – the tip of the electric field intensity describes a circle
3. Elliptical polarization – the tip of the electric field intensity describes an ellipse
4. Rotation: Circularly and elliptically polarized waves can rotate clockwise or counterclockwise as they propagate. Counterclockwise rotation is said to be **right elliptically (or circularly) polarized** because it follows the right hand rule – the thumb is in the direction of propagation and the curled fingers show the direction of rotation of the electric field. **Left circularly (or elliptically) polarized** waves rotate clockwise as they propagate towards the observer.