Summary Chapter 11

The main topic in this chapter is the introduction of displacement current density in Ampere's law and its consequences. The final result is Maxwell's equations which include the postulates in the previous chapters but also the modification due to displacement currents. The displacement current density modifies Ampere's law by adding the term $\mathbf{J}_d = \partial\mathbf{D}/\partial t$ [A/m^2] as follows

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \qquad (11.6)$$

This, together with Faraday's and Gauss's laws, form what are called **Maxwell's equations**

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad (11.24) \qquad \text{or:} \qquad \oint_C \mathbf{E}.d\mathbf{l} = -\frac{\partial\Phi}{\partial t} \quad (11.28)$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \quad (11.25) \qquad \text{or:} \qquad \oint_C \mathbf{H}.d\mathbf{l} = \int_s\left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right)\cdot d\mathbf{s} \quad (11.29)$$

$$\nabla\cdot\mathbf{D} = \rho_v \quad (11.26) \qquad \text{or:} \qquad \oint_s \mathbf{D}\cdot d\mathbf{s} = Q \quad (11.30)$$

$$\nabla\cdot\mathbf{B} = 0 \quad (11.27) \qquad \text{or:} \qquad \oint_s \mathbf{B}\cdot d\mathbf{s} = 0 \quad (11.31)$$

The materials constitutive relations $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, and the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ are part of the general system of equations called the Maxwell-Lorentz equations (see **Table 11.1**).
Time dependent potentials are defined based on the properties of the curl and divergence of fields:

$\mathbf{E} = -\nabla V$, if $\nabla\times\mathbf{E} = 0$ (11.38), V is the electric scalar potential (voltage)
$\mathbf{H} = -\nabla\psi$, if $\nabla\times\mathbf{H} = 0$ (11.39), ψ is the magnetic scalar potential
$\mathbf{B} = \nabla\times\mathbf{A}$, because $\nabla\cdot\mathbf{B} = 0$ (11.40), $\mathbf{A}$ is the magnetic vector potential

The time-dependent electric field intensity, based on Ampere's law (**Eq. (11.24)**) is:

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla V \quad \left[\frac{\text{V}}{\text{m}}\right] \qquad (11.45)$$

Gauges define the divergence of vector potentials (in this case the magnetic vector potential)
$\nabla\cdot\mathbf{A} = 0$ for static fields (Coulomb's gauge)
$\nabla\cdot\mathbf{A} = -\mu\varepsilon\frac{\partial V}{\partial t}$ (11.49) for time dependent fields (Lorenz's gauge)

Interface conditions for time dependent fields are identical to those for static fields as discussed in Chapters 4 and 9. These are summarized in **Tables 11.3, 11.4** and **11.5** (See **Fig. 11.2** for reference):

$$E_{1t} = E_{2t}, \quad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \quad \text{and} \quad D_{1n} - D_{2n} = \rho_s, \quad \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$$

$$\hat{\mathbf{n}}\times(\mathbf{H}_{1t} - \mathbf{H}_{2t}) = \mathbf{J}_s, \quad \hat{\mathbf{n}}\times\left(\frac{\mathbf{B}_{1t}}{\mu_1} - \frac{\mathbf{B}_{2t}}{\mu_2}\right) = \mathbf{J}_s \quad \text{and} \quad B_{1n} = B_{2n}, \quad \mu_1 H_{1n} = \mu_2 H_{2n}$$

Electromagnetic fields are often representsed in terms of phasors. **Phasor representation** of any function A, (scalar or vector) is as follows:

$$A_p(x,y,z) = A_0(x,y,z)e^{j\theta} = A_0(x,y,z)\angle\theta = A_0(x,y,z)\cos\theta + jA_0(x,y,z)\sin\theta \qquad (11.65)$$

Transformation into the time-domain

$$A(x,y,z,t) = \mathrm{Re}\{A_0(x,y,z)e^{j\theta}e^{j\omega t}\} \qquad (11.66)$$

$$\frac{d}{dt}(A(x,y,z,t)) = \mathrm{Re}\{j\omega A_\rho(x,y,z)e^{j\omega t}\} \qquad (11.67)$$

Time harmonic field equations play an important role in electromagnetics. Maxwell's equations in the frequency domain (see **Table 11.6**) are:

$$\nabla\times\mathbf{E} = -j\omega\mathbf{B} \quad (11.68) \qquad \text{or:} \qquad \oint_C \mathbf{E}.d\mathbf{l} = -j\omega\int_s \mathbf{B}.d\mathbf{s} \quad (11.72)$$

$$\nabla\times\mathbf{H} = \mathbf{J} + j\omega\mathbf{D} \quad (11.69) \qquad \text{or:} \qquad \oint_C \mathbf{H}\cdot d\mathbf{l} = \int_s (\mathbf{J} + j\omega\mathbf{D})\cdot d\mathbf{s} \quad (11.73)$$

$$\nabla\cdot\mathbf{D} = \rho_v \quad (11.70) \qquad \text{or:} \qquad \oint_s \mathbf{D}\cdot d\mathbf{s} = Q \quad (11.74)$$

$$\nabla\cdot\mathbf{B} = 0 \quad (11.71) \qquad \text{or:} \qquad \oint_s \mathbf{B}\cdot d\mathbf{s} = 0 \quad (11.75)$$

where **E, H, D, B**, and **J** are vector phasors and Q and ρ_v are scalar phasors. Note however that we do not mark these in any particular way – it is understood from the context when these quantities must be phasors.

Source-free equations are obtained by setting $\mathbf{J} = 0$, $\rho_v = 0$ in either the time- or frequency-domain equations.