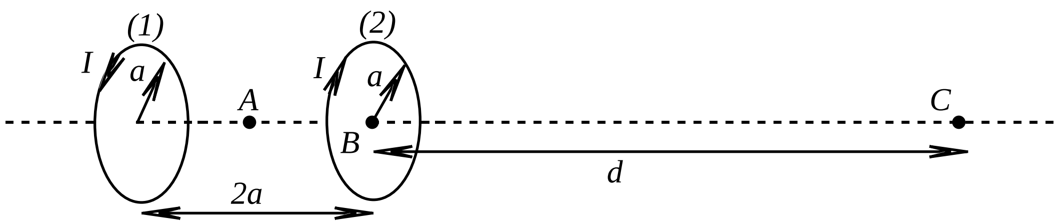
304-351A, Electromagnetic Fields
Final Exam
December 14, 2001
Solution

1. e9.8. In chapter9.problems.extra. Two loops, made of thin wire carry equal and opposite currents as shown in the figure below. The radius of each loop is *a* and the distance between the loops is *2a*. The loops are in free space and placed parallel to each other, so that they are both perpendicular to the axis *A-C* as shown (that is, like two wheels on an axle). Calculate:
a. The magnetic flux density at point *A* (midway between the two loops).
b. The magnetic flux density at point *B* (at the center of loop No. *2*).
c. The magnetic flux density at point *C* if *d>>a*.
Note: In all three cases give both direction and magnitude of the magnetic flux density.



Solution: Calculate the magnetic flux density on the axis, due to a loop. This was calculated in **example 8.3**. The solution at any point on the axis is then the superposition of the fields of the two loops. The latter is given as:

$$|\mathbf{H}| = \frac{Ia^2}{2(a^2 + h^2)^{3/2}} \quad \rightarrow \quad |\mathbf{B}| = \frac{\mu_0 I a^2}{2(a^2 + h^2)^{3/2}}$$

The solution is written here as magnitudes because the magnetic flux density depends on the direction of the current. We will use the right hand rule to identify these directions.

a. *B=0*. The loops are identical and at equal distance from A. The magnetic flux densities they produce at A cancel each other.

b. Loop (1), produces a magnetic flux density pointing to the right (this is taken as positive). With h=2a:

$$B_A = \frac{\mu_0 I a^2}{2(a^2 + h^2)^{3/2}} = \frac{\mu_0 I a^2}{2(a^2 + 4a^2)^{3/2}} \quad [\mathrm{T}]$$

$$\boxed{B_A = \frac{\mu_0 I a^2}{2(a^2 + 4a^2)^{3/2}} \quad [\mathrm{T}]}$$

Loop (2) produces a magnetic flux density pointing to the left (negative). With h=0:

$$\boxed{B_B = -\frac{\mu_0 I a^2}{2(a^2)^{3/2}} \quad [\mathrm{T}]}$$

The total field at B is now:

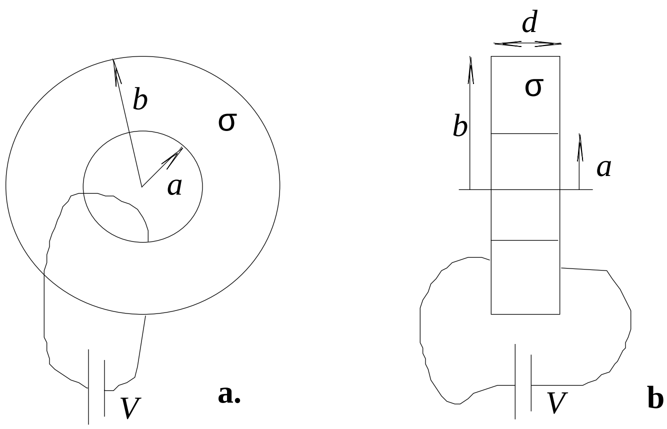
$$B_t = B_A - B_B = \frac{\mu_0 I a^2}{2(a^2 + 4a^2)^{3/2}} - \frac{\mu_0 I a^2}{2(a^2)^{3/2}} = \frac{\mu_0 I}{a}\left(\frac{1}{10\sqrt{5}} - \frac{1}{2}\right) = -0.455\frac{\mu_0 I}{a}$$

$$\boxed{B_t = \frac{\mu_0 I a^2}{2(a^2 + 4a^2)^{3/2}} - \frac{\mu_0 I a^2}{2(a^2)^{3/2}} = -0.455\frac{\mu_0 I}{a} \qquad [\mathrm{T}]}$$

The magnetic flux density points to the left.

c. At large distances the magnetic field intensities of the two loops is approximately the same and in opposite directions. Thus, $\boxed{B_C = 0}$.

2. e7.10. In chapter7.problems.extra. A resistor is made in the form of a conducting circular washer with inner radius equal to *a* and outer radius equal to *b*. The thickness of the washer is equal to *d*. If the conductivity of the conductor is σ, calculate the total resistance of the washer:
a. If the source V is connected between the inner and outer surfaces (as shown in (**a**)).
b. If the source is connected between the two flat surfaces as in (**b**).



Solution:
a.

$$R = \frac{V}{I} \qquad I = \frac{V}{R} \qquad J = \frac{V}{R2\pi rd}$$

$$\mathbf{J} = \sigma\mathbf{E} \qquad E = \frac{J}{\sigma} \qquad E = \frac{V}{R\sigma 2\pi rd}$$

$$V = \int_a^b \mathbf{E}.\mathbf{dl} = \frac{V}{R\sigma 2\pi d}\int_a^b \frac{1}{r}\,dr$$

Thus the resistance is

$$R = \frac{1}{\sigma 2\pi d}\int_a^b \frac{1}{r}\,dr = \frac{1}{\sigma 2\pi d}\ln\frac{b}{a}$$

$$R = \frac{1}{\sigma 2\pi d}\ln\frac{b}{a} \quad [\Omega]$$

b.

$$S = \pi(b^2 - a^2) \qquad I = \frac{V}{R} \qquad J = \frac{V}{R\pi(b^2 - a^2)}$$

$$E = \frac{V}{\sigma R\pi(b^2 - a^2)}$$

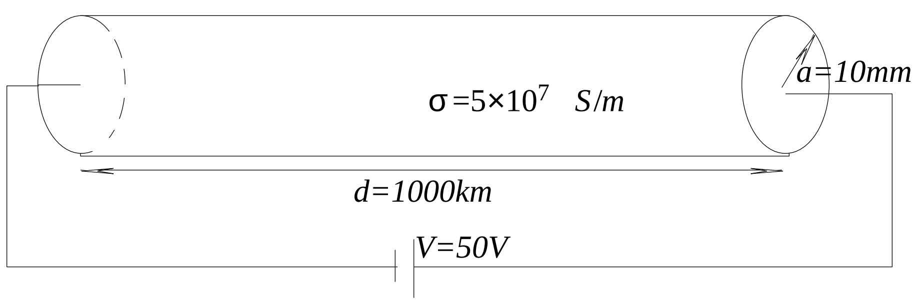
and is uniform. Thus,

$$V = Ed = \frac{Vd}{\sigma R\pi(b^2 - a^2)}$$

and the resistance is

$$R = \frac{d}{\sigma\pi(b^2 - a^2)} \quad [\Omega]$$

3. e8.4. in chapter8.problems.extra). A cylindrical conductor of radius a=10mm is made of copper with conductivity $\sigma=5\times10^7$ S/m. The conductor is used for one conductor of a power line that is d=1000 *km* long. Because of the current in the conductor, a potential drop of V=50V exists across the conductor (see Figure). Calculate the magnetic field intensity everywhere in space, including in the interior of the conductor. Sketch the magnetic field intensity.



Solution: To find the magnetic field intensity we must first calculate the current in the conductor and to do so we must have the resistance.

The resistance of a cylindrical conductor of length d and radius a is:

$$R = \frac{d}{\sigma\pi a^2} \quad [\Omega]$$

The current in the conductor is now:

$$I = \frac{V}{R} = \frac{V\sigma\pi a^2}{d} \qquad [A]$$

Now we distinguish two regions. One is *r>a*, the second is *0<r≤a*.

For *0<r≤a*, we use Ampere's law by drawing a contour of radius *r<a*. This contour encloses an area πr^2 while the current is uniform in an area πa^2. Thus, the total current enclosed by the contour is:

$$I_r = I\frac{\pi r^2}{\pi a^2} = I\frac{r^2}{a^2} = \frac{V\sigma\pi r^2}{d} \qquad [A]$$

The length of the contour is *2πr* and we can write:

$$2\pi r H = I_r = \frac{V\sigma\pi r^2}{d} \qquad \rightarrow \qquad \boxed{H = \frac{V\sigma r}{2d} \quad \left[\frac{\text{A}}{\text{m}}\right]}$$

or:

$$H(r \le a) = \frac{50\times 5.0\times 10^7 r}{2\times 10^6} = 1250r \qquad \left[\frac{\text{A}}{\text{m}}\right]$$

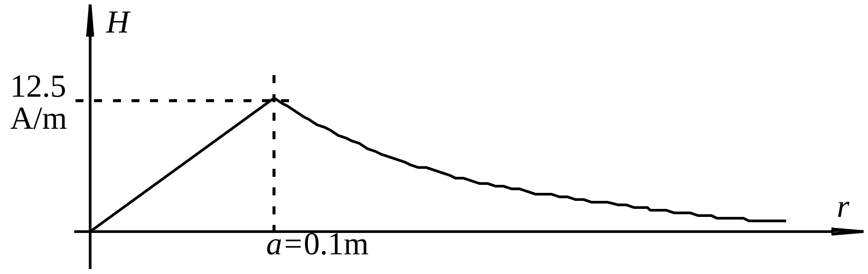
For *r>a*, all current is enclosed by a the contour and we can write:

$$2\pi r H = I = \frac{V\sigma\pi a^2}{d} \qquad \rightarrow \qquad \boxed{H = \frac{V\sigma a^2}{2dr} \quad \left[\frac{\text{A}}{\text{m}}\right]}$$

or:

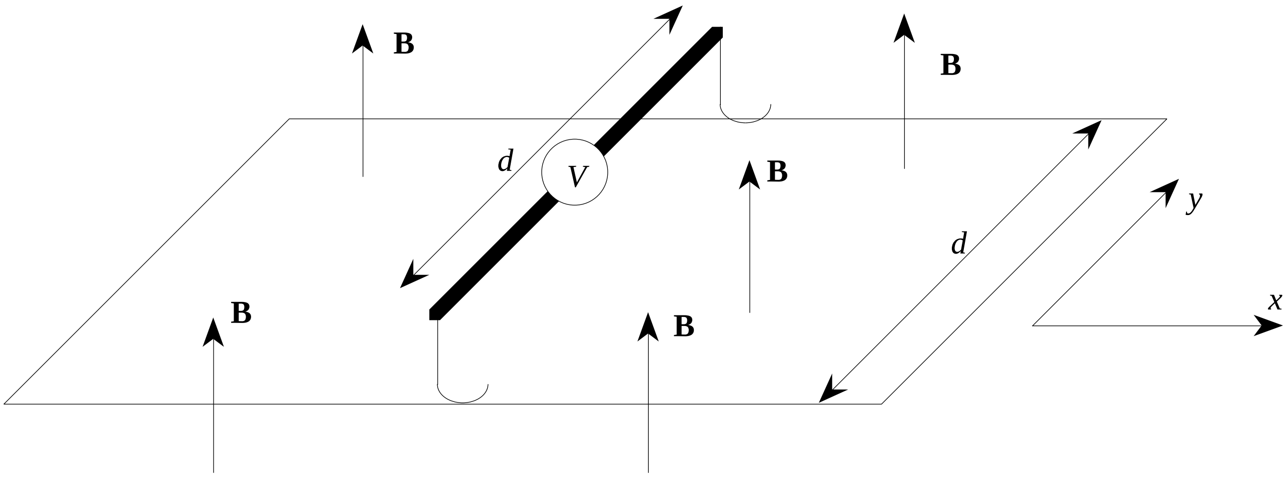
$$H(r>a) = \frac{50\times 5.7\times 10^7\times (0.01)^2}{2\times 10^6\times r} = \frac{0.125}{r} \qquad \left[\frac{\text{A}}{\text{m}}\right]$$

At *r*=0, the magnetic field intensity is zero. At *r=a* it is equal to 14.25 A/m. Between zero and a the magnetic field intensity grows linearly. After that it diminishes as *1/r*. The plot of the field with distance from the center of the conductor is shown in the figure below.



e10.4. In chapter10.problems.extra. A very long sheet of metal of width *d* is placed in a uniform, perpendicular magnetic field as shown. A voltmeter is connected to the two opposite sides with stiff wires as shown. Suppose the contacts of the voltmeter can slide on the sheet.
What is the emf read by the voltmeter if:

a. The voltmeter together with the connecting wires is moved at a velocity v_x in the x direction.
b. If the sheet is moved in the y direction at a velocity v_y and the voltmeter is sliding in the x direction at a velocity (while still keeping the contacts, that is, it is also sliding in the y direction with the plate) v_x.
c. If the sheet is moved at a velocity v_x in the x direction and the voltmeter is stationary with respect to the sheet.



a. The emf is: $emf = Bdv_x$
The bar moves in a magnetic field.

b. Motion in the y direction produces no emf on the bar, but the motion in the x direction does:

$emf = Bdv_x$

c. emf=0. The bar and conductor form a loop. A loop moving in a constant magnetic field produces zero emf. Another way to look at it is that the loop formed by the bar is parallel to the flux density. Thus, zero flux passes through the loop. Note: there is a potential difference between the two edges of the stiff wire but the voltmeter is now connected in a loop and the total voltage it measures is zero.

e10.40. (in chapter10.exams.extra). A transformer is made as shown. The thickness of the transformer is d and the cross-sectional area of the transformer is the same everywhere and equal to S. The permeability of the core is large but not infinite. Two coils each having N turns are wound; coil (1) is wound on the left "leg" of the transformer; coil (2) is wound on the central leg of the transformer. A current $I_1 = A\sin\omega t$ passes through coil (1), calculate the ratio between the induced voltage (emf) in coil (2) and coil (1).

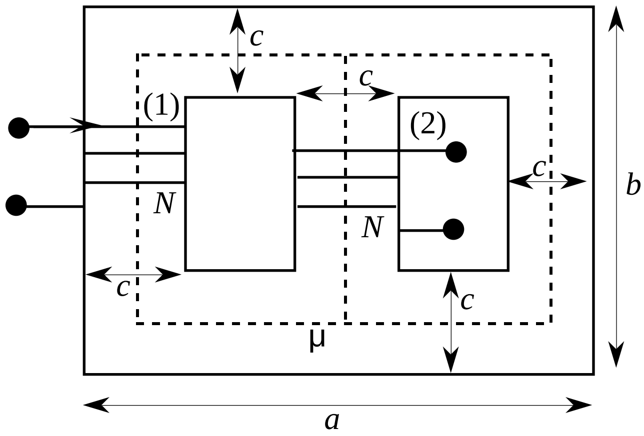


Figure A

Note: the thickness (into the page) of the transformer core is given as *d*.

Solution: First, we draw the equivalent magnetic circuit as follows:

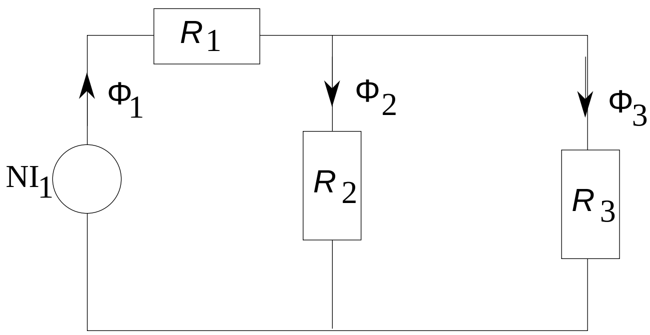


Figure B

The emfs in the two coils are, by definition:

$$|emf_1| = N\frac{d\Phi_1}{dt}, \qquad |emf_2| = N\frac{d\Phi_2}{dt}$$

Thus, we need to calculate the fluxes Φ_1 and Φ_2. From the equivalent circuit we write:

$$\Phi_1 = \frac{NI_1}{R_1 + (R_2 \| R_3)}, \qquad \Phi_2 = \frac{NI_1}{R_1 + (R_2 \| R_3)} \cdot \frac{(R_2 \| R_3)}{R_2}$$

where $R_2 \| R_3$ means that R_2 and R_3 are parallel to each other. Now, we can calculate the emfs:

$$|emf_1| = N\frac{d\Phi_1}{dt} = N\frac{N(dI_1/dt)}{R_1 + (R_2 \| R_3)}, \qquad |emf_2| = N\frac{d\Phi_2}{dt} = N\frac{N(dI_1/dt)}{R_1 + (R_2 \| R_3)} \cdot \frac{(R_2 \| R_3)}{R_2}$$

The ratio is:

$$\frac{emf_2}{emf_1} = \frac{N\frac{N(dI_1/dt)}{R_1 + (R_2 \| R_3)} \cdot \frac{(R_2 \| R_3)}{R_2}}{N\frac{N(dI_1/dt)}{R_1 + (R_2 \| R_3)}} = \frac{(R_2 \| R_3)}{R_2}$$

or:

$$\frac{emf_2}{emf_1} = \frac{(R_2 \| R_3)}{R_2} = \frac{R_2 R_3}{R_2(R_2 + R_3)} = \frac{R_3}{R_2 + R_3}$$

Now we calculate the reluctances (average paths are shown in Figure A.

$$R_1 = R_3 = \frac{a+b-2c}{\mu cd}, \qquad R_2 = \frac{b-c}{\mu cd}$$

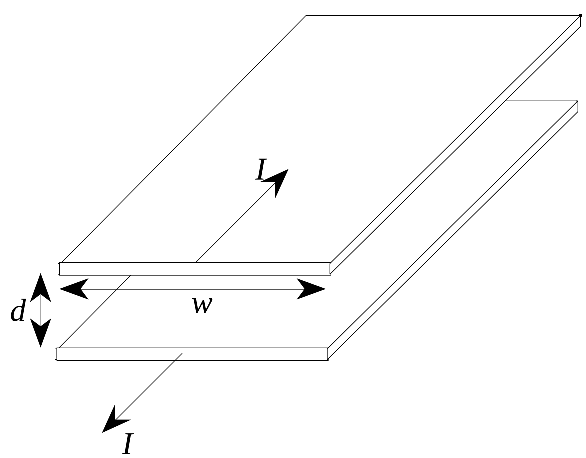
Thus:

$$\frac{emf_2}{emf_1} = \frac{R_3}{R_2+R_3} = \frac{\dfrac{a+b-2c}{\mu cd}}{\dfrac{a+b-2c}{\mu cd} + \dfrac{b-c}{\mu cd}} = \frac{a+b-2c}{a+2b-3c}$$

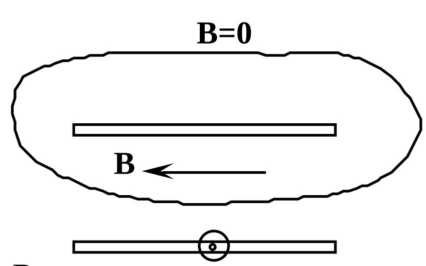
$$\boxed{\frac{emf_2}{emf_1} = \frac{a+b-2c}{a+2b-3c}}$$

You could have argued as follows: The flux Φ_1 splits into 2 equal parts, only one of these parts (half the flux) linking with coil (2). Thus the mutual inductance between the two coils is half of what it would be if all flux linked. Thus, the emf in coil (2) must be half of what it would be if all flux links. Since the number of turns is the same in both coils, the emf in coil (2) must be half the emf in coil (1).

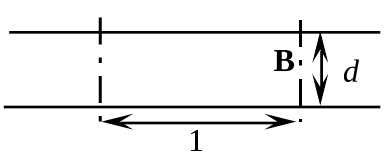
e9.4. (in chapter9.exams.extra). Two very thin conducting sheets are w=100mm wide and are separated a distance d=1mm apart. A current I=20A passes through the sheets, uniformly distributed on the width of the sheets. Assume w>>d and calculate:
a. The inductance per unit length of the system of two sheets.
b. The force per unit length acting on the sheets. Indicate the direction of the force on each of the sheets.



A.



B



C.

a. Calculate B: since w>>d, the whole assembly is like parallel plates. From the loop above (Figure B):

$$Bw = \mu_0 I \qquad \rightarrow \qquad B = \frac{\mu_0 I}{w} \qquad [T]$$

To calculate inductance we need the flux per unit length of the assembly (see Figure C):

$$\Phi = B \times 1 \times d = \frac{\mu_0 I d}{w} \qquad [wb]$$

The flux linkage equals the flux (one turn only) thus,

$$L_{11} = \frac{\Phi}{I} = \frac{\mu_0 d}{w} = \frac{4\pi \times 10^{-7} \times 10^{-3}}{0.1} = 1.256 \times 10^{-8} \qquad \left[\frac{H}{m}\right]$$

$$\boxed{L_{11} = \frac{\mu_0 d}{w} = 1.256 \times 10^{-8} \qquad \left[\frac{\mathrm{H}}{\mathrm{m}}\right]}$$

b. Force. The energy in the gap between the two conductors is:

$$w_b = \frac{B^2}{2\mu_0} \qquad \left[\frac{J}{m^3}\right]$$

If the gap is changed by a distance *dl*, the change in energy in the gap and force on the conductors are:

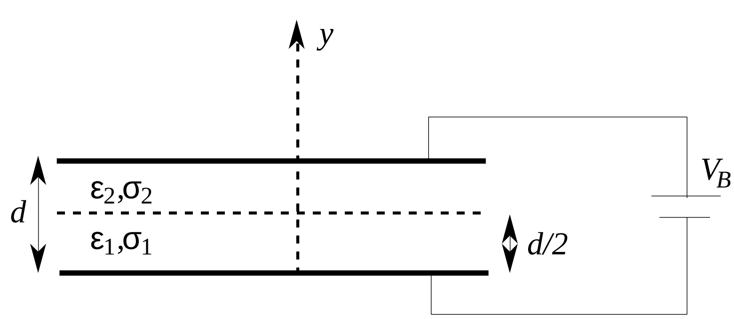
$$dW = w_b dv = w_b \times w \times 1 \times dl \qquad \rightarrow$$

$$F = \frac{dW}{dl} = w_b w = \frac{B^2 w}{2\mu_0} = \frac{\mu_0 I^2}{2w} = \frac{4\pi \times 10^{-7} \times 400}{2 \times 0.1} = 0.0025 \qquad \left[\frac{N}{m}\right]$$

$$\boxed{|F| = \frac{\mu_0 I^2}{2w} = 0.0025 \qquad \left[\frac{\mathrm{N}}{\mathrm{m}}\right]}$$

The two plates tend to separate (you can see that from the right hand rule since $d\mathbf{f} = \mathbf{J} \times \mathbf{B}$ and this must be perpendicular to both current and flux density)

e7.28 In chapter7.problems.extra. A capacitor is made of two parallel plates of area *A* separated by a distance *d*. A battery is connected holding the lower plate at ground potential and the upper plate at potential V_B. Filling the space between the plates are two lossy dielectric slabs each of thickness $d/2$. The permittivities and conductivities of the dielectrics are given by ε_1, ε_2, σ_1, and σ_2.



Neglecting any edge effects, the potential distribution between the plates may be found from Laplace's equation with the appropriate boundary conditions as

$$V = \frac{2\sigma_2 V_B}{d(\sigma_1 + \sigma_2)} y, \qquad 0<y<d/2$$

$$V = \frac{2\sigma_1 V_B}{d(\sigma_1 + \sigma_2)} y + \frac{(\sigma_2 - \sigma_1)V_B}{(\sigma_1 + \sigma_2)}, \qquad d/2<y<d$$

a. Find the electric field intensity in each dielectric region.
b. Find the current I and the power P supplied by the battery.

Solution: Calculate the electric field intensity as the negative gradient of the potential:
a.

$$\mathbf{E}_1 = -\nabla V_1 = -\hat{\mathbf{y}}\frac{2\sigma_2 V_B}{d(\sigma_1 + \sigma_2)}, \qquad \mathbf{E}_2 = -\nabla V_2 = -\hat{\mathbf{y}}\frac{2\sigma_1 V_B}{d(\sigma_1 + \sigma_2)} \qquad \left[\frac{\text{V}}{\text{m}}\right]$$

b.

$$\mathbf{J}_1 = \sigma_1 \mathbf{E}_1 = -\hat{\mathbf{y}}\frac{2\sigma_1\sigma_2 V_B}{d(\sigma_1 + \sigma_2)}, \qquad \mathbf{J}_2 = \sigma_2 \mathbf{E}_2 = -\hat{\mathbf{y}}\frac{2\sigma_1\sigma_2 V_B}{d(\sigma_1 + \sigma_2)} = \mathbf{J}_1$$

$$I = |\mathbf{J}|A = \frac{2\sigma_1\sigma_2 A V_B}{d(\sigma_1 + \sigma_2)} \qquad [\text{A}]$$

$$P = V_B I = \frac{2\sigma_1\sigma_2 A V_B^2}{d(\sigma_1 + \sigma_2)} \qquad [\text{W}]$$

e8.79 In chapter8.problems.extra 8. Two concentric, very thin spherical-shell conductors have radii a and b, $a<b$. The space between the shells is filled with a dielectric medium with perrnittivity ε. A battery is connected so that the outer shell is held at potential $V(b)=V_b$, and the inner one at $V(a)$=0.
a. Find the potential and the electric field intensity at all points between the shells.
b. Find the total charge on each shell.

Solution:
a. Start with Laplace's Eq. in Cylindrical coordinates (field can only vary with r so we can write:

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) = 0$$

Integrating once:

$$R^2\frac{\partial V}{\partial R} = C_1 \qquad \rightarrow \qquad \frac{\partial V}{\partial R} = \frac{C_1}{R^2}$$

Integrating again,

$$V = -\frac{C_1}{R} + C_2$$

The potential is zero at $R=a$:

$$0 = -\frac{C_1}{a} + C_2$$

The potential is V_b at $R=b$:

$$V_b = -\frac{C_1}{b} + C_2$$

Substituting:

$$V_b = -\frac{C_1}{b} + \frac{C_1}{a} = C_1\left(\frac{b-a}{ab}\right)$$

Thus:

$$C_1 = \frac{abV_b}{b-a}, \qquad C_2 = \frac{bV_b}{b-a}$$

and the general solution is:

$$\boxed{V(R) = \frac{bV_b}{b-a}\frac{R-a}{R} \quad [\mathrm{V}]}$$

The electric field intensity is:

$$\mathbf{E}(R) = -\nabla V(R) = -\hat{\mathbf{R}}\frac{abV_b}{b-a}\left(\frac{1}{R^2}\right)$$

Thus:

$$\boxed{\mathbf{E}(R) = -\hat{\mathbf{R}}\frac{abV_b}{(b-a)R^2} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]}$$

b. From Gauss' law:

$$|Q_a| = 4\pi a^2\varepsilon E(a) = \frac{4\pi a^2\varepsilon abV_b}{(b-a)a^2} = \frac{4\pi\varepsilon abV_b}{(b-a)} \quad [\mathrm{C}]$$

$$Q_b = -4\pi b^2\varepsilon E(b) = -\frac{4\pi b^2\varepsilon abV_b}{(b-a)b^2} = -\frac{4\pi\varepsilon abV_b}{(b-a)} = -Q_a \quad [\mathrm{C}]$$

But, because the charge is negative on the inner shell (since the electric field intensity points in the negative R direction), we have:

$$\boxed{Q_a = -\frac{4\pi\varepsilon abV_b}{(b-a)} \quad [\mathrm{C}] \qquad Q_b = \frac{4\pi\varepsilon abV_b}{(b-a)} \quad [\mathrm{C}]}$$