Electromagnetics I
Final Exam
December 15, 2000
Solution

Solve the following 5 problems. Each problem is 20% of the grade. To receive full credit, you must show all work. If you need to assume anything, state your assumptions clearly. Reasonable assumptions that are necessary to solve the problem will be accepted. In all problems assume properties of free space ($\varepsilon_0=8.85\times10^{-12}$, $\mu_0=4\pi\times10^{-7}$).

1. e9.75. (In Chapter9.problems.extra). Two very small loops are located as shown below. Assume $h>>a$, $h>>b$. Each carries a current I. The loops are arranged so that their centers are on the same axis and their areas parallel to each other. Calculate the change in the total magnetic energy stored in the system (which consists of the two coils), if the distance between the loops is changed to $2h$.

Solution: Mark the upper loop as (2) and the lower loop as (1). Now, using **Eq. (9.11)** we calculate the flux density due to loop (1) at loop (2) and the corresponding fluxes. From this we can calculate the mutual inductance between the coils and the total energy due to mutual inductance. Since, as we move the coils further apart, only the mutual inductance changes, there is no need to calculate the self inductance (which is very difficult to calculate). The difference between the energy stored in the mutual inductance gives the change in the magnetic energy.

The flux densities are as follows (using **Eq. (9.11)** with θ=0):

$$\mathbf{B}_{12}=\hat{\mathbf{z}}\frac{\mu_0 I b^2}{2h^3} \qquad \mathbf{B}_{12}=\hat{\mathbf{z}}\frac{\mu_0 I a^2}{2h^3} \qquad [\mathrm{T}]$$

The fluxes are:

$$\Phi_{12}=\frac{\mu_0 I b^2}{2h^3}\pi a^2 \qquad \Phi_{12}=\hat{\mathbf{z}}\frac{\mu_0 I a^2}{2h^3}\pi b^2 \qquad [\mathrm{Wb}]$$

Thus, the mutual inductances are:

$$L_{12}=L_{21}=\frac{\mu_0 b^2\pi a^2}{2h^3} \qquad [\mathrm{H}]$$

The energy with the loops a distance h apart is:

$$W_1=2\frac{L_{12}I^2}{2}=L_{12}I^2=\frac{\mu_0 b^2\pi a^2 I^2}{2h^3} \qquad [\mathrm{J}]$$

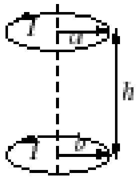
Now, replacing h by 2h, we get:

$$W_2=\frac{\mu_0 b^2\pi a^2 I^2}{2(2h)^3}=\frac{\mu_0 b^2\pi a^2 I^2}{16h^3} \qquad [\mathrm{J}]$$

The change in energy is:

$$\Delta W = W_2 - W_1 = \frac{\mu_0 b^2 \pi a^2 I^2}{16h^3} - \frac{\mu_0 b^2 \pi a^2 I^2}{2h^3} = -\frac{7\mu_0 \pi b^2 a^2 I^2}{16h^3} \qquad [\text{J}]$$

Note that this energy is performed by the system indicating a reduction in the stored energy.



2. **e8.35. (In Chapter8.problems.extra).** The following vector field is given.
a. Show that this field cannot be a magnetic field.
b. Show that this field cannot be an electric field.

$$\mathbf{P} = \hat{\mathbf{x}} yx + \hat{\mathbf{y}} xy$$

Solution: Apply the postulates of the magnetic and electric field intensity. If these are not satisfied in a general sense, the field cannot be magnetic or electric.

a. The postulates of the magnetic field are as follows:

$$\times \mathbf{H} = \mathbf{J} \qquad\qquad .\mathbf{B} = 0$$

Thus, we calculate the curl and the divergence of **P**:

$$\times \mathbf{P} = \hat{\mathbf{z}}\left(\frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y}\right) = \hat{\mathbf{z}}(y - x) \neq 0$$

$$.\mathbf{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} = y + x \neq 0$$

Thus, because the divergence is not zero, this cannot be a magnetic field.

b. For the electric field we require:

$$\times \mathbf{E} = 0 \qquad\qquad .\mathbf{D} = \rho$$

Since the curl calculated in (a) is nonzero, this cannot be an electric field.

3. **e7.21. (In Chapter7.problems.extra).** Two perfectly conducting, very thin, spherical shells are given as shown. The space between the shells is filled with a conducting material with conductivity σ and the shells are connected in a circuit as shown. Calculate the current in the circuit.

Solution: Define an element of the resistance of thickness *dr* and cross sectional area of $4\pi r^2$ as shown in **Figure B**. Integrate from $r=a$ to $r=b$ to obtain the total resistance of this spherical resistor. Divide voltage by resistance to obtain current.

The resistance of the element shown is:

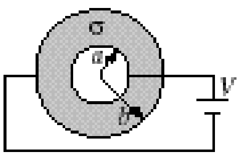
$$dR = \frac{dl}{\sigma S} = \frac{dr}{4\pi\sigma r^2} \qquad [\Omega]$$

The total resistance is:

$$R = \int_{r=a}^{r=b} \frac{dr}{4\pi\sigma r^2} = -\frac{1}{4\pi\sigma r}\bigg|_a^b = \frac{1}{4\pi\sigma}\left[\frac{1}{a} - \frac{1}{b}\right] \qquad [\Omega]$$

The current is:

$$I = \frac{V}{R} = R = \frac{4\pi\sigma V}{\left[\frac{1}{a} - \frac{1}{b}\right]} = \frac{4\pi\sigma V ab}{b - a} \qquad [\mathrm{A}]$$



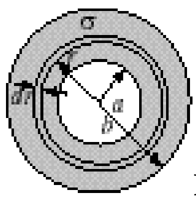


Figure B

4. **e9.76. (In Chapter9.problems.extra).** A parallel plate capacitor is given . The capacitor has plate area of 1m^2 and the distance between the plates is 0.1mm. The space between the plates is filled with a dielectric of relative permittivity 4 and the capacitor is connected to a 10V battery (see **Figure A**). Suppose now that the dielectric is pulled sideways so that the dielectric only fills half the space between the plates as shown in **Figure B**. Calculate the amount of work needed to pull the dielectric from the condition in **Figure A** to that in **Figure B**.

Solution: Calculate the capacitance in each configuration and, given the voltage, the energy stored in each configuration. The difference between the energies gives the amount of work needed to move the dielectric.

The capacitance in **Figure A** is marked as C_1:

$$C_1 = \frac{4\varepsilon_0 S}{d} \qquad [\mathrm{F}]$$

In Figure B, we have two capacitors in parallel. Each has the same plate separation but only half the plate area. One of these has air as the dielectric, the second, has a dielectric. Thus:

$$C_2 = C_a + C_d = \frac{\varepsilon_0 S/2}{d} + \frac{4\varepsilon_0 S/2}{d} = \frac{5\varepsilon_0 S}{2d} \qquad [\mathrm{F}]$$

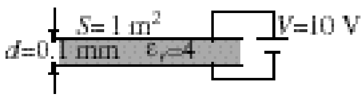
The energies are:

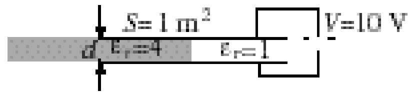
$$W_1 = C_1\frac{V^2}{2} = \frac{4\varepsilon_0 S V^2}{2d} \qquad [\mathrm{J}]$$

$$W_2 = C_2\frac{V^2}{2} = \frac{5\varepsilon_0 S V^2}{4d} \qquad [\mathrm{J}]$$

Thus:

$$\Delta W = W_2 - W_2 = \frac{5\varepsilon_0 S V^2}{4d} - \frac{4\varepsilon_0 S V^2}{2d} = -\frac{3\varepsilon_0 S V^2}{4d} = -\frac{3\times 8.854\times 10^{-12}\times 1\times 10^2}{4\times 10^{-4}} = -6.64\times 10^{-6} \qquad [\mathrm{J}]$$





5. **e8.36. (In Chapter8.problems.extra).** Two very thin conducting sheets carry current densities as shown in the figure below. The upper sheet carries a current density J_1 [A/m] flowing into the page. The lower sheet carries a current density J_2 [A/m], flowing out of the page. A thin insulating layer is placed between the two sheets. Assuming the sheets to be very large (essentially infinite) and the current density to be uniform, calculate:
a. The magnetic field intensity outside the sheets (above and below)
b. The magnetic field intensity in the insulating sheet.

Solution: Use ampere's law by drawing arbitrary contours as shown in **Figure B**. The total field in each domain is the sum of the fields of the two sheets.

a. Due to the upper sheet, using the contour marked as (a), we get:

$$2H_1L = J_1L \quad \rightarrow \quad H_1 = \frac{J_1}{2} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

Note that the directions indicate the following:

$$\mathbf{H} = \hat{\mathbf{x}}\frac{J_1}{2} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \qquad \text{above upper plate}$$

$$\mathbf{H} = -\hat{\mathbf{x}}\frac{J_1}{2} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \qquad \text{below upper plate}$$

Due to the lower plate, using contour marked as (b), we have:

$$\mathbf{H} = -\hat{\mathbf{x}}\frac{J_2}{2} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \qquad \text{above lower plate}$$

$$\mathbf{H} = \hat{\mathbf{x}}\frac{J_2}{2} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \qquad \text{below lower plate}$$

Now, summing these up we have outside the plates:

$$\mathbf{H} = \hat{\mathbf{x}}\frac{J_1}{2} - \hat{\mathbf{x}}\frac{J_2}{2} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \qquad \text{above upper plate}$$

$$\mathbf{H} = \hat{\mathbf{x}}\frac{J_2}{2} - \hat{\mathbf{x}}\frac{J_1}{2} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \qquad \text{below lower plate}$$

b. Between the plates, we sum the field below the upper plate and the field above the lower plate

$$\mathbf{H} = -\hat{\mathbf{x}}\frac{J_1}{2} - \hat{\mathbf{x}}\frac{J_2}{2} \qquad \left[\frac{\text{A}}{\text{m}}\right]$$

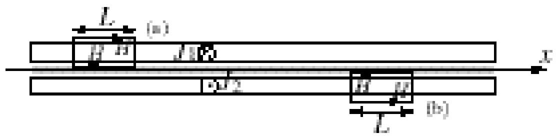


Figure B