Electromagnetics I
Exam No. 3
December 1, 2003
Solution

Please read the exam carefully. Solve the following 4 problems. Each problem is 1/4 of the grade. To receive full credit, you must show all work. If I cannot understand what you are doing, you will receive no credit. If you need to assume anything, state your assumptions clearly. Reasonable assumptions that are necessary to solve the problem will be accepted. In all problems assume properties of free space (ε_0=8.85×10^{-12} F/m, μ_0=4π×10^{-7} H/m) unless otherwise specified. You can write on both sides of the page. If you need additional space ASK for additional paper and make sure you write your name on it.

Level of difficulty: 2 (most difficult), 3, 4, 1 (easiest)

e9.166. 1. In Chapter9.problems.extra 5.
a. Which of the following fields are possible magnetic fields in free space?

1. $\mathbf{B}_1 = \hat{\mathbf{x}}2xy + \hat{\mathbf{y}}(2z + 3) + \hat{\mathbf{z}}(5 - 2yz)$
2. $\mathbf{B}_2 = \hat{\mathbf{x}}(x^2 + y^2) + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}xz$
3. $\mathbf{B}_3 = \hat{\mathbf{x}}(9 - x^3) + \hat{\mathbf{y}}(z^3 - x^3) + \hat{\mathbf{z}}(x^3 - y^3)$

b. For the possible magnetic fields in question , determine the current density **J.**
Note: A guess, even if correct, will not be accepted.

Solution: To show that a field is a magnetic field we need to show that its divergence is zero and its curl is non-zero. That is, we apply the postulates:

$$\nabla\times\mathbf{H} = \mathbf{J}, \qquad \nabla.\mathbf{B} = 0$$

However, **J** can be zero so it is important to calculate the divergence first.

a. We first calculate the divergence. By definition we must have:

$$\nabla.\mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

For the four fields:

$$\nabla.\mathbf{B}_1 = \frac{\partial(2xy)}{\partial x} + \frac{\partial(2z+3)}{\partial y} + \frac{\partial(5-2yz)}{\partial z} = 2y - 2y = 0$$

$$\nabla.\mathbf{B}_2 = \frac{\partial(x^2+y^2)}{\partial x} + \frac{\partial(3)}{\partial y} - \frac{\partial(xz)}{\partial z} = 2x - x \neq 0$$

$$\nabla.\mathbf{B}_3 = \frac{\partial(9-x^3)}{\partial x} + \frac{\partial(z^3-x^3)}{\partial y} + \frac{\partial(x^3-y^3)}{\partial z} = -3x^2 \neq 0$$

Thus, only $\mathbf{B}_1$ can be a magnetic field.
(b). The curl of B is:

$$\nabla\times\mathbf{H} = \mathbf{J} \quad \rightarrow \quad \nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

Thus:

$$\nabla\times\mathbf{B}_1 = \mu_0\mathbf{J} = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$= \hat{\mathbf{x}}\left(\frac{\partial(5-2yz)}{\partial y} - \frac{\partial(2z+3)}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial(2xy)}{\partial z} - \frac{\partial(5-2yz)}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial(2z+3)}{\partial x} - \frac{\partial(2xy)}{\partial y}\right)$$

$$= -\hat{\mathbf{x}}(2z+2) - \hat{\mathbf{z}}2x$$

Therefore, the current density is:

$$\mathbf{J} = -\hat{\mathbf{x}}\frac{(2z+2)}{\mu_0} - \hat{\mathbf{z}}\frac{2x}{\mu_0} \qquad \left[\frac{\text{A}}{\text{m}^2}\right]$$

e8.91. In Chapter8.problems.extra. 2. Two circular, conducting cylinders lie with their axes at $x=+c$ and $x=-c$ $(c<b/2)$ and each of radius a $(2a<b)$ and each carries a uniformly distributed current $I/2$ in the positive z-direction (out of the page). Surrounding these is a thin, circular cylindrical shell with its axis at $x=0$ and radius b, carrying a uniformly distributed current I in the reverse direction (into the page). **Figure A** shows the configuration in cross-section. the solid cylinders and the shells may be assumed to be infinitely long. Calculate the magnetic field intensity (direction and magnitude) at any point inside the outer cylinder but outside the smaller cylinders.

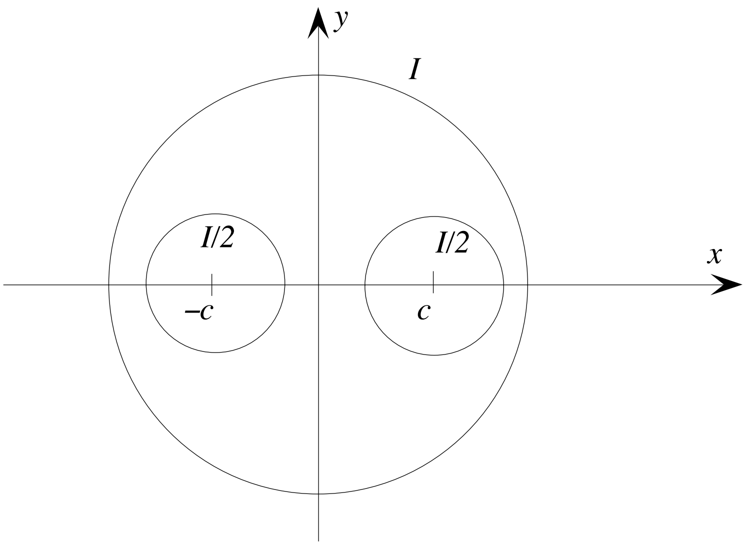

Figure A.

Solution: Use Ampere's law since the cylinders are infinitely long.

a. First, we note that the outer shell produces no field inside itself. The field is therefore only due to the inner cylinders. Taking a general point (x,y) outside the inner cylinders we have the configuration in **Figure B** for the left cylinder. This cylinder produces a magnetic field intensity as follows:

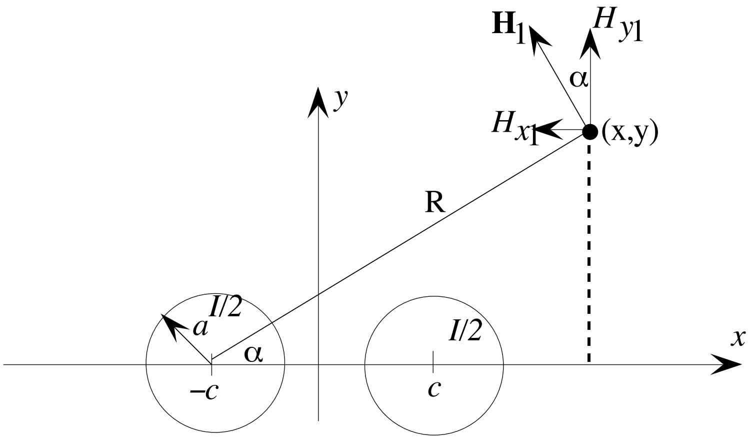

Figure B.

$$H_1 = \frac{I/2}{2\pi R} = \frac{I}{4\pi R} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

The x and y components are:

$$H_{x1} = \frac{I}{4\pi R} sin\alpha = \frac{I}{4\pi R}\frac{y}{R} = \frac{Iy}{4\pi R^2} = \frac{Iy}{4\pi\left((x+c)^2 + y^2\right)} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

$$H_{y1} = \frac{I}{4\pi R} cos\alpha = \frac{I}{4\pi R}\frac{(x+c)}{R} = \frac{I(x+c)}{4\pi R^2} = \frac{I(x+c)}{4\pi\left((x+c)^2 + y^2\right)} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

In vector form we have:

$$\mathbf{H}_1 = -\hat{\mathbf{x}}\frac{Iy}{4\pi\left((x+c)^2 + y^2\right)} + \hat{\mathbf{y}}\frac{I(x+c)}{4\pi\left((x+c)^2 + y^2\right)} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

Repeating the process for the right hand side cylinder (see **Figure C**):

$$H_2 = \frac{I/2}{2\pi R} = \frac{I}{4\pi R} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

$$H_{x2} = \frac{I}{4\pi R} sin\alpha = \frac{I}{4\pi R}\frac{y}{R} = \frac{Iy}{4\pi R^2} = \frac{Iy}{4\pi\left((x-c)^2 + y^2\right)} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

$$H_{y2} = \frac{I}{4\pi R} cos\alpha = \frac{I}{4\pi R}\frac{(x-c)}{R} = \frac{I(x-c)}{4\pi R^2} = \frac{I(x-c)}{4\pi\left((x-c)^2 + y^2\right)} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

In vector form:

$$\mathbf{H}_2 = -\hat{\mathbf{x}}\frac{Iy}{4\pi\left((x-c)^2 + y^2\right)} + \hat{\mathbf{y}}\frac{I(x-c)}{4\pi\left((x-c)^2 + y^2\right)} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

The total field is:

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$$

$$= -\hat{\mathbf{x}}\frac{Iy}{4\pi}\left[\frac{1}{\left((x-c)^2 + y^2\right)} + \frac{1}{\left((x+c)^2 + y^2\right)}\right] + \hat{\mathbf{y}}\frac{I}{4\pi}\left[\frac{(x-c)}{\left((x-c)^2 + y^2\right)} + \frac{(x+c)}{\left((x+c)^2 + y^2\right)}\right] \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

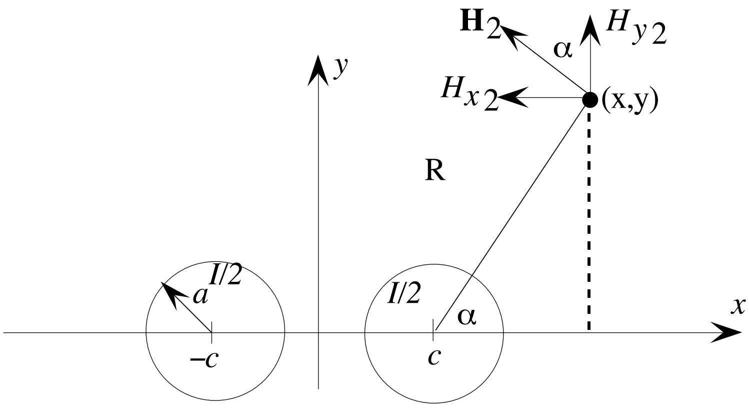

Figure C.

3. **e9.55. (From probl.tosolve).** Calculate the inductance per unit length of the parallel plate conductors shown. Assume that the current in the conductors is equal to *I* amperes, uniformly distributed and that the plates are very thin. Assume $b \ll w$.

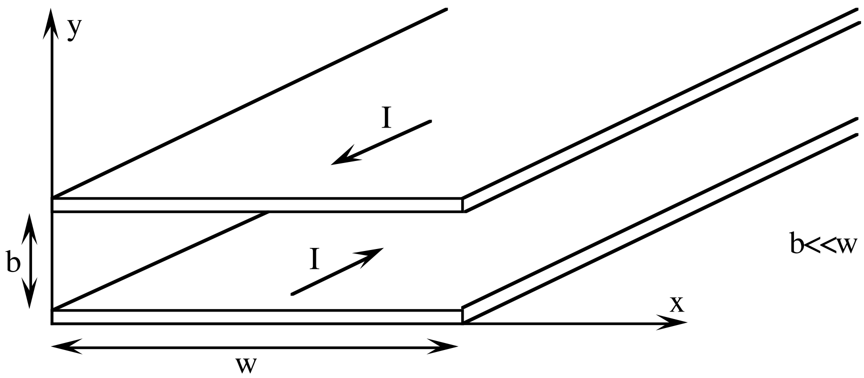

Figure A.

Solution: Since the distance between the plates is very small, we may assume that the field between the plates is the same as for infinitely wide plates (see example 8.8). Thus, using Ampere's law for, say, the upper plate, we get (see **Figure B**):

$$H = \frac{I}{2w}$$

The magnetic field intensity due to the lower plate points in the same direction as in Figure B and is also equal to I/2w. Thus, the magnetic field intensity between the plates is:

$$H = \frac{I}{w}$$

The magnetic flux, which crosses the area parallel to the y-z plane is (for a length in the z direction equal to 1m:

$$\Phi = \mu_0 HS = \mu_0 \frac{I}{w} 1 \times b = \frac{\mu_0 Ib}{w}$$

The inductance per unit length is therefore:

$$L = \frac{\Phi}{I} = \frac{\mu_0 b}{w} \qquad \left[\frac{\text{H}}{\text{m}}\right]$$

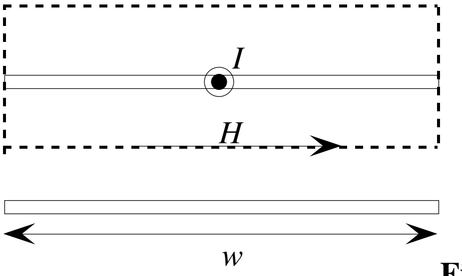

Figure B.

e9.104 From Bastos.scan.additional. 4.4. An infinite wire and a square loop are given as shown in **Figure A**. The loop and wire are in the same plane (see figure) but there is no electrical contact between the two (that is, the wire sits on top of the loop without actually touching it). Calculate the mutual inductance between the wire and loop for the configuration shown.

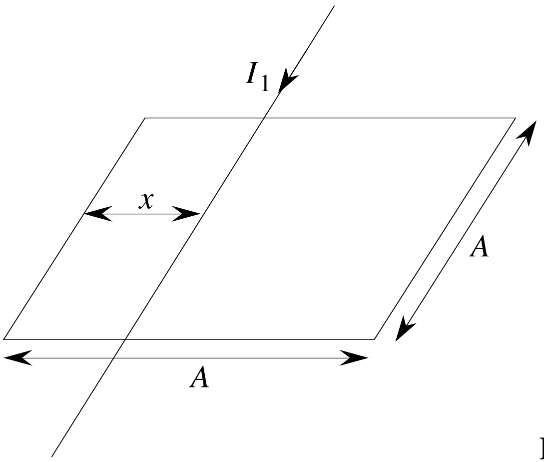

Figure A

Solution: Using **example 9.11** as a guide, we solve for the magnetic flux for the general configuration in **Figure A**. Assuming an arbitrary current *I* in the wire, the magnetic flux density at a radial distance *r* from the wire is:

$$B = \frac{\mu_0 I}{2\pi r} \qquad [\text{T}]$$

We note (see **Figure B**), that the flux to the right of the wire points up while the flux to the left points down. The total net flux through the loop is the difference between these two fluxes. Taking an element of area as shown, we get for the flux to the right of the wire:

$$\Phi_1 = \int_{r=0}^{r=A-x} \frac{\mu_0 I}{2\pi r} A dr = \frac{\mu_0 IA}{2\pi} [\ln r]_0^{A-x}$$

The flux to the left of the wire is:

$$\Phi_2 = \int_{r=0}^{r=x} \frac{\mu_0 I}{2\pi r} A dr = \frac{\mu_0 IA}{2\pi}\left[ln\ r\right]_0^x$$

Calculating the difference:

$$\Phi = \Phi_1 - \Phi_2 = \frac{\mu_0 IA}{2\pi}\left[ln\ r\right]_0^{A-x} - \frac{\mu_0 IA}{2\pi}\left[ln\ r\right]_0^x = \frac{\mu_0 IA}{2\pi}\left[ln\ (A-x) - ln\ (0) - ln\ (x) + ln\ (0)\right]$$
$$= \frac{\mu_0 IA}{2\pi}\left[ln\ (A-x) - ln\ (0) - ln\ (x) + ln\ (0)\right] = \frac{\mu_0 IA}{2\pi}\ ln\frac{A-x}{x}$$

Note: normally you would not write ln(0) but I wanted to show that this term cancels out due to the subtraction and so the fact that it is not defined does not mean that a solution does not exist.

Since this is the total flux that links the wire and loop we can calculate the mutual inductance by dividing the total flux by the current in the wire. Thus:

$$L_{12} = \frac{\mu_0 A}{2\pi}\ ln\frac{A-x}{x} \qquad [\mathrm{H}]$$

Note that this makes sense since, for $x=A/2$, $L_{12}=0$ as required.

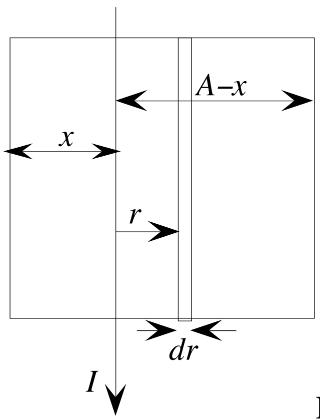

Figure B.