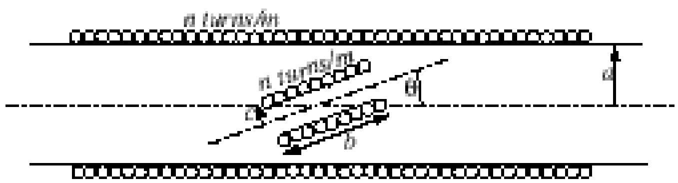
Electromagnetics I
Exam No. 3
December 4, 2000

Solve the following 4 problems. Each problem is 25% of the grade. To receive full credit, you must show all work. If you need to assume anything, state your assumptions clearly. Reasonable assumptions that are necessary to solve the problem will be accepted. In all problems assume permittivity of free space is ($\varepsilon_0 = 8.854\times10^{-12}$ F/m and permeability of free space is $4\pi\times10^{-7}$ H/m).
Level of difficulty: 3 (most difficult), 4, 1, 2 (easiest).

1. A very long solenoid (you may assume it is infinite) of radius *a* [m] has *n* turns per unit length. A short solenoid is *b* [m] long, has radius *c* [m] and also has *n* turns per unit length. The two solenoids are arranged as in the figure below. Assuming that $b<2a$ (so that the small solenoid may be rotated inside the long solenoid), calculate the mutual inductance between the short and long solenoids. Assume permeability of free space everywhere.



Solution: Assume a current *I* in the large solenoid and calculate the flux density in the long solenoid. Now calculate the flux in the small solenoid produced by the large solenoid. Multiply by the number of turns in the small solenoid to obtain the flux linkage. Divide by the assumed current to find the mutual inductance.

With a current I in the long solenoid (the solenoid is marked here as “1”) the flux density inside the solenoid is:

$$B_1 = \mu_0 n I \qquad [\mathrm{T}]$$

Because the axis of the small solenoid (marked here as ‘2’), is at an angle θ to the axis of the large solenoid, the flux through the small solenoid is:

$$\Phi_{12} = \int_{s2} \mathbf{B}_1 . d\mathbf{s}_2 = B_1 S_2 \cos\theta = \mu_0 n I \pi c^2 \cos\theta \qquad [\mathrm{Wb}]$$

The flux linkage is:

$$\Lambda_{12} = nb\Phi_{12} = \mu_0 n^2 b I \pi c^2 \cos\theta \qquad [\mathrm{Wb.t}]$$

Note that the total number of turns in the short solenoid is *nb*. The inductance is therefore:

$$L_{12} = \mu_0 n^2 b \pi c^2 \cos\theta \qquad [\mathrm{H}]$$

2. A very large sheet made of a conducting material is shown in the figure below. A uniform current density *J* [A/m^2] exists in the sheet, flowing into the page. Assume permeability of free space everywhere.

a. Calculate the magnetic flux density outside the sheet.
b. Calculate the magnetic flux density inside the sheet.
c. What is the flux density on the center line of the sheet?

Solution: Because the current distribution is flat, we generate a symmetric contour (symmetric about the horizontal axis) of arbitrary width L and of thickness equal to twice the distance to the point at which we need the magnetic flux density.

a. Outside the current distribution, we have the situation in **Figure B** (rightmost contour). Now the contour, which is 2y thick and L wide, only includes a current in a section which is 2b thick. Thus, using the same relation as before, we write:

$$\mu_0 \int_C \mathbf{H}.d\mathbf{l} = \mu_0 I_{enclosed} \quad \rightarrow \quad 2\mu_0 HL = \mu_0 JL(2b) \quad \rightarrow \quad \mu_0 H = B = \mu_0 Jb$$

Thus:

$$\mathbf{B} = \hat{\mathbf{x}}\mu_0 Jb \qquad b<y<\infty \qquad \mathbf{B} = -\hat{\mathbf{x}}\mu_0 Jb \qquad -\infty<y<-b \qquad [\mathrm{T}]$$

b. Figure B shows the appropriate contour as well as the directions of the magnetic flux density above and below the axis. Integrating around the contour we get:

$$\mu_0 \int_C \mathbf{H}.d\mathbf{l} = \mu_0 I_{enclosed} \quad \rightarrow \quad 2\mu_0 HL = \mu_0 JL(2y) \quad \rightarrow \quad \mu_0 H = B = \mu_0 Jy$$

Since the field for y>0 is in the positive x direction and for y<0 in the negative y direction, we write:

$$\mathbf{B} = \hat{\mathbf{x}}\mu_0 Jy \qquad 0<y<b \qquad \mathbf{B} = -\hat{\mathbf{x}}\mu_0 Jy \qquad -b<y<0 \qquad [\mathrm{T}]$$

c. The answer is zero. You can argue either that in (a), y=0 or, alternatively, that the total current enclosed is zero.

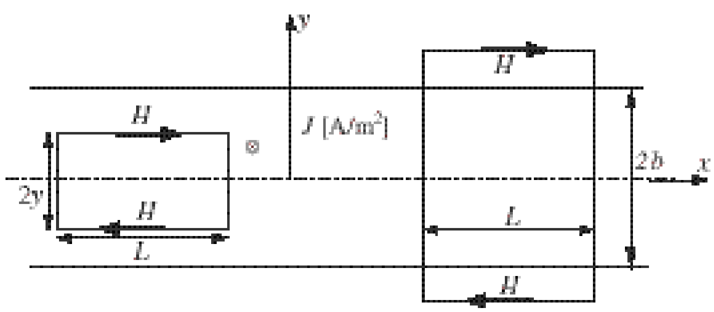


Figure B.

3. A thick conductor of radius *a* [m], carries a current *I* [A]. This current produces a magnetic flux density B_0 at a point *P*, a distance *b* [m] from the center of the conductor (see **Figure A**). Because of overheating in the conductor it is proposed to drill two holes so that the conductor may be cooled using water. If the magnetic flux density at point *P* in **Figure B** must remain the same as in **Figure A**, what must be the current the conductor must now carry? That is, obtain an expression for the current in **Figure B** in terms of B_0 (given)

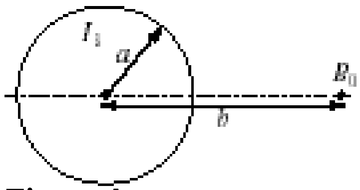


Figure A

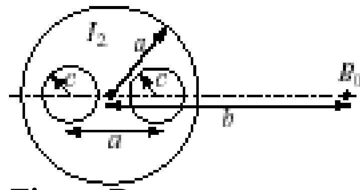


Figure B

Solution: Use Ampere's law. First, assume a current density $J=I_2/(\pi a^2-2\pi c^2)$ exists in the whole of the conductor, disregarding the two holes. Calculate the field at *P*. Now calculate the fields of two conductors, identical in dimensions and location to the two holes with current density equal to $J=I_2/(\pi a^2-2\pi c^2)$ but flowing in the direction opposite I_2. The sum of these three fields at *P* must equal B_0.

The field intensity at *P* due to a conductor of radius *a* carrying a current density $J=I_2/(\pi a^2-2\pi c^2)$ is:

$$B_1 = \frac{\mu_0 \pi a^2 \left[I_2/\left(\pi a^2 - 2\pi c^2\right)\right]}{2\pi b} = \frac{\mu_0 \pi a^2 I_2}{2\pi b\left(\pi a^2 - 2\pi c^2\right)} \quad [\mathrm{T}]$$

Suppose that this filed points up at *P* (that is, I_2 points out of the page).
Taking now the hole on the left, the total current we must have in the "hole" must be:

$$I_{hl} = -J\pi c^2 = \frac{\pi c^2 I_2}{\pi a^2 - 2\pi c^2} \quad [\mathrm{A}]$$

The distance from the center of this hole to point *P* is $b+a/2$ and we have:

$$B_{hl} = \frac{\mu_0 I_{hl}}{2\pi(b+a/2)} = -\frac{\mu_0 c^2 I_2}{\left(a^2 - 2c^2\right)(2b+a)} \quad [\mathrm{T}]$$

Similarly, for the hole on the right, we have exactly the same current (the two holes are identical) but the distance to *P* is $b-a/2$. We get:

$$B_{hr} = -\frac{\mu_0 I_2 c^2}{\left(a^2 - 2c^2\right)(2b-a)} \quad [\mathrm{T}]$$

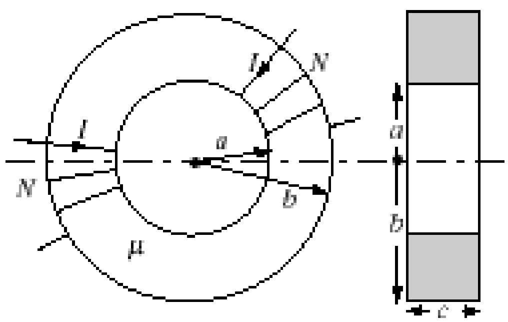
Both of these fields point downwards (if we assume that B_1 points upwards). Thus, the total field is:

$$B_0 = B_1 + B_{hl} + B_{hr} = \frac{\mu_0 a^2 I_2}{2\pi b\left(a^2 - 2c^2\right)} - \frac{\mu_0 c^2 I_2}{\pi\left(a^2 - 2c^2\right)(2b+a)} - \frac{\mu_0 I_2 c^2}{\pi\left(a^2 - 2c^2\right)(2b-a)} = \frac{\mu_0 I_2}{2\pi}\left(\frac{a^2}{\left(a^2 - 2c^2\right)b} - \frac{2c^2}{\left(a^2 - 2c^2\right)(2b+a)} - \frac{2c^2}{\left(a^2 - 2c^2\right)(2b-a)}\right) \quad [\mathrm{T}]$$

Thus, the necessary current to produce a flux density B_0 is:

$$I_2 = \frac{2\pi B_0}{\mu_0\left(\frac{a^2}{\left(a^2 - 2c^2\right)b} - \frac{2c^2}{\left(a^2 - 2c^2\right)(2b+a)} - \frac{2c^2}{\left(a^2 - 2c^2\right)(2b-a)}\right)} \quad [\mathrm{A}]$$

4. A torus is wound with two identical coils, each having N turns uniformly distributed around the torus, and each carrying a current I [A]. The inner radius of the torus is a [m], the outer radius is b [m], its thickness is c [m], and the permeability of the torus is μ [H/m]. Calculate:
a. The minimum possible energy stored in the torus.
b. The maximum possible energy stored in the torus.
Note: The directions of the currents shown in the figure below are given for illustration purposes.



Solution: First calculate the two self inductances and the two mutual inductances. The energy stored is then calculated from the inductance. For minimum stored energy, the two mutual inductances must be negative. This occurs when the fluxes of the two coils are in the same direction. For maximum stored energy, the two mutual inductances must be positive.

a. The magnetic flux density in a torus is found by postulating a contour of radius $a<r<b$ which encloses N turns (for each coil). Thus, for coil (1), we can write:

$$B_1 = \frac{\mu N_1 I_1}{2\pi r} = \frac{\mu NI}{2\pi r} \quad [\mathrm{T}]$$

The flux due to coil (1) is found by integrating this over the cross sectional area of the torus:

$$\Phi_{11} = \int_s \mathbf{B}.d\mathbf{s} = \int_r Bcdr = \int_{r=a}^{b} \frac{\mu NIc}{2\pi r} = \frac{\mu NIc}{2\pi}\, ln\,(b/a) \quad [\mathrm{Wb}]$$

This flux links all the turns of coil (1). Therefore, we can write the flux linkage and the self inductance of this coil as:

$$\Lambda_{11} = \frac{\mu N^2 Ic}{2\pi} \, ln \, (b/a) \qquad and \qquad L_{11} = \frac{\mu N^2 c}{2\pi} \, ln \, (b/a) \qquad [\text{H}]$$

Since coil (2) is identical to coil (1) in all respects, we have:

$$L_{22} = \frac{\mu N^2 c}{2\pi} \, ln \, (b/a) \qquad [\text{H}]$$

The mutual inductance is found as follows: Since we have the flux produced by either coil, we argue that Φ_{11} links all the turns of coil (2). Thus, $\Lambda_{12}=\Phi_{11}N_2$. This gives:

$$\Lambda_{12} = \frac{\mu N^2 Ic}{2\pi} \, ln \, (b/a) \qquad and \qquad L_{12} = \frac{\mu N^2 c}{2\pi} \, ln \, (b/a) \qquad [\text{Wb.t}]$$

Similarly:

$$L_{21} = L_{21} = \frac{\mu N^2 c}{2\pi} \, ln \, (b/a) \qquad [\text{H}]$$

Note: In this case, because the two coils are idential and all flux links all turns, the four inductances (L_{11}, L_{22}, L_{12} and L_{21}) are identical.

The energy stored in the torus is:

$$W = L \frac{I^2}{2} = \frac{L_{11}I^2}{2} + \frac{L_{22}I^2}{2} \pm \frac{L_{12}I^2}{2} \pm \frac{L_{21}I^2}{2} \qquad [\text{J}]$$

Minimum energy occurs when the energy due to mutual inductances subtracts from the energy due to self inductances. In this case:

$$W_{min} = 0$$

b. To obtain maximum inductance, all four terms must be positive. In this case:

$$W_{max} = 4 \frac{1}{2} \frac{\mu N^2 c I^2}{2\pi} \, ln \, (b/a) = \frac{\mu N^2 I^2 c}{\pi} \, ln \, (b/a) \qquad [\text{J}]$$