Electromagnetics I
Exam No. 3
July 30, 2001
Solution

Solve the following 5 problems. Each problem is 20% of the grade. To receive full credit, you must show all work. If you need to assume anything, state your assumptions clearly. Reasonable assumptions that are necessary to solve the problem will be accepted. In all problems assume properties of free space ($\varepsilon_0=8.854\times10^{-12}$ F/m, $\mu_0=4\pi\times10^{-7}$ H/m).

1. A wire bent as shown in **Figure A** carries a current *I*. Calculate the magnetic flux density at point *P* (midway on the vertical line). Assume the wire is in free space.

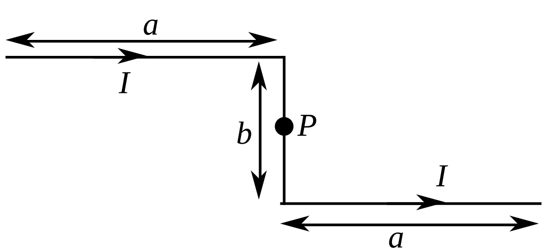

Figure A

Solution: This problem may be solved by inspection; since the upper an lower conductors produce opposite fields at *P*, these cancel each other (the upper and lower segments are equal in size). The field of the vertical segment is also zero since the field of any filament, at its own center is zero. Thus the magnetic flux density at *P* is zero.

2. Two tori (1 torus, 2 tori) of identical dimensions (inner radius is *a*, outer radius is *b*, and the thickness of each torus is *c*, are placed side by side (see **Figure** A) and are wound as follows: Coil (1), has N_1 turns and is wound on torus (1). Coil (2) has N_2 turns and is wound on torus 2. Coil 3 is wound around both tori and has N_3 turns. The permeability of each torus is μ and you may assume *(b-a)>>a*. Calculate:

a. The mutual inductance between coil (1) and (2).
b. Calculate the mutual inductance between coil (2) and (3)
c. The mutual inductance between coil (1) and (3)

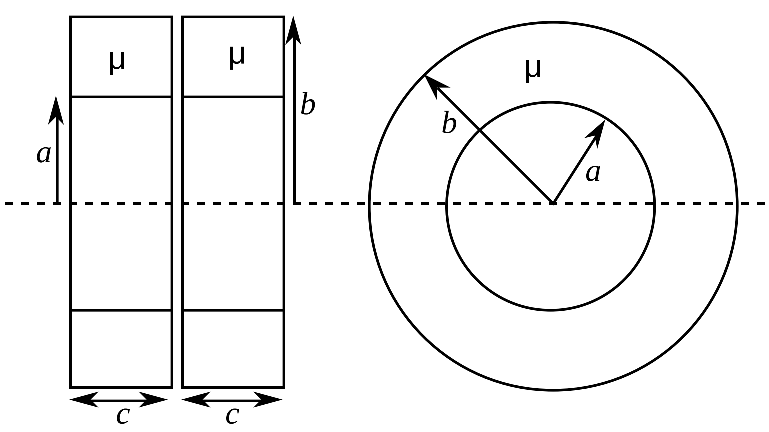

Figure A

Solution: Calculate the magnetic flux density and magnetic flux that would be produced by each torus. Coils (1) and (2) only link flux with themselves, producing a self inductance but no mutual inductance. Coil 3 links with either the flux in coil (1) or coil (2) producing nonzero mutual inductance.

Assuming a current I in coil (1) we have the flux density as:

$$B_1 = \frac{\mu N_1 I_1}{2\pi r} = \frac{\mu N_1 I_1}{2\pi(a+b)/2} = \frac{\mu N_1 I_1}{\pi(a+b)}$$

where $r=(a+b)/2$ was used as the average radius of the torus. The flux in torus (1) is:

$$\Phi_1 = B_1 S = \frac{\mu N_1 I_1}{\pi(a+b)}(b-a)c = \frac{\mu N_1 I_1 (b-a)c}{\pi(a+b)}$$

Similarly:

$$\Phi_2 = B_2 S = \frac{\mu N_2 I_2}{\pi(a+b)}(b-a)c = \frac{\mu N_2 I_2 (b-a)c}{\pi(a+b)}$$

a. The flux in coil (1) only passes through coil (1) or coil (3). Thus, $L_{12}=0$.
b. The flux in coil (2), links (passes through) all N3 loops of coil (3). Thus:

$$\Lambda_{23} = \Phi_2 N_3 = \frac{\mu N_2 N_3 I_2 (b-a)c}{\pi(a+b)}$$

and the mutual inductance between coil (2) and (3) is:

$$L_{23} = \frac{\Lambda_{23}}{I_2} = \frac{\mu N_2 N_3 (b-a)c}{\pi(a+b)}$$

c. Similarly, the flux in coil (1) passes through coil (3):

$$L_{13} = \frac{\Lambda_{23}}{I_1} = \frac{\mu N_1 N_3 (b-a)c}{\pi(a+b)}$$

3. A solenoid has radius *a* and *n* turns per unit length. A very small loop of radius *b* ($b<<a$) is placed inside the solenoid as shown in **Figure A**. The axis of the solenoid and that of the loop make an angle α. Calculate the mutual inductance between loop and solenoid. The material in the solenoid is free space.

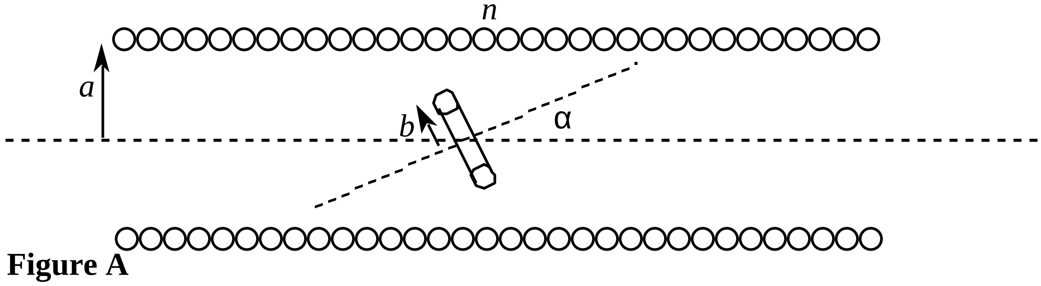

Figure A

Solution: Calculate the flux density of the solenoid and then the flux that links the solenoid and the loop. Divide by the (assumed) current I in the solenoid to find the mutual inductance.

Assuming a current I in the solenoid, the flux density in the solenoid is:

$$B = \mu_0 n I$$

Since only that part of the flux density that is normal to the loop links the loop, the flux in the loop is:

$$\Phi = \int_{s_{loop}} \mathbf{B}.d\mathbf{s}_{loop} = \mu_0 n I \pi b^2 cos\alpha$$

The loop has one turn so this is also the flux linkage. The mutual inductance is therefore:

$$L_{12} = \frac{\Phi}{I} = \mu_0 n \pi b^2 cos\alpha$$

4. A very long power line is made of two wires, separated a distance *d* apart as shown in **Figure A**. The power line carries a current *I*. A square loop is placed midway between the two wires of the power line and has dimensions as shown. Calculate the flux in the loop.

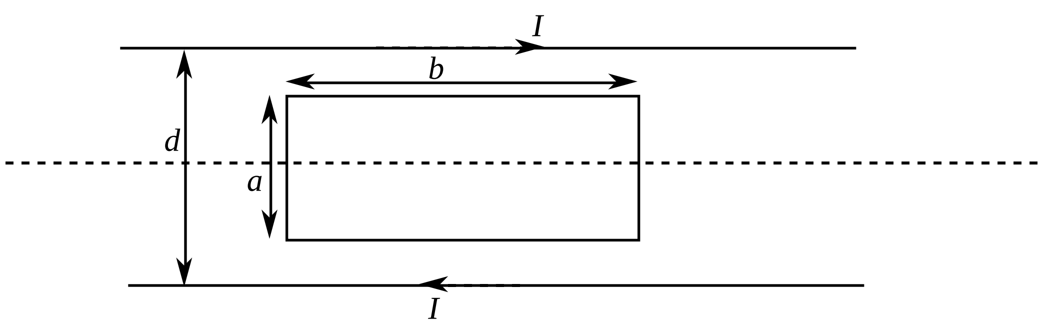

Figure A

Solution: Taking first the upper conductor, its flux density at a distance *r* is:

$$B_1 = \frac{\mu_0 I}{2\pi r}$$

The flux in the loop is calculated with the aid of **Figure B**:

$$\Phi_1 = \int_{r=(d-a)/2}^{(d+a)/2} \frac{\mu_0 I}{2\pi r} b dr = \frac{\mu_0 I b}{2\pi} ln\left(\frac{d+a}{d-a}\right)$$

Since the lower conductor ptoduces identical flux (and in the same direction) as the upper conductor, the total flux is:

$$\Phi = \frac{\mu_0 I b}{\pi} ln\left(\frac{d+a}{d-a}\right)$$

5. The magnetic vector potential of a system of conductors in space is given in cylindrical coordinates as

$$\mathbf{A} = \hat{\mathbf{r}} z^2 \varphi + \hat{\mathbf{z}} z^2 r$$

Calculate the magnetic field intensity everywhere.

The magnetic flux density is calculated from $\times \mathbf{A}$ in cylindrical coordinates:

$$\mathbf{B} = \quad \times \mathbf{A} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \hat{\varphi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial (r A_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right)$$

$$= \hat{\varphi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left(-\frac{\partial A_r}{\partial \varphi}\right) = \hat{\varphi}(2z\varphi - z^2) - \hat{\mathbf{z}}\frac{z^2}{r}$$

Since we need the magnetic field intensity, we write:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \hat{\varphi}\frac{(2z\varphi - z^2)}{\mu_0} - \hat{\mathbf{z}}\frac{z^2}{\mu_0 r}$$