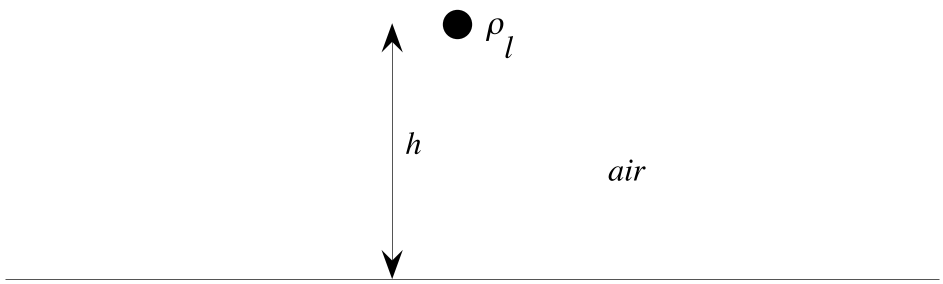
Electromagnetics I
Exam No. 2
July 16, 2004
Solutions

Solve the following 4 problems. Each problem is 25% of the grade. To receive full credit, you must show all work. If you need to assume anything, state your assumptions clearly. Reasonable assumptions that are necessary to solve the problem will be accepted. In all problems assume properties of free space (ε_0=8.854×10^{-12} F/m).

e5.30. In chapter5.problems.extra. **1.** An infinitely long wire is placed at a height *h* above a perfectly conducting plane as shown in cross section in the figure below. The wire carries a line charge density of ρ_l [C/m]. Calculate the potential midway between the wire and the plane, directly below the wire. Assume permittivity of free space.



Solution: The system above may be replaced by an image line (negative) together with the original line. The system looks as in the figure below. Now, using the solution for a line of charge, we calculate the electric field due to the upper line at a general point above the location of the conductor. Then we integrate from the surface of the conductor (which is at zero potential) to h/2. The total fielde is the sum of the fields of the two lines of charge since the two fields are in the same direction (pointing down). Note that Gauss's law is used implicitly.

Due to upper line of charge at a height y':

$$E^+ = \frac{\rho_l}{2\pi\varepsilon_0 r} = \frac{\rho_l}{2\pi\varepsilon_0 (h - y')}$$

Due to the lower line of charge:

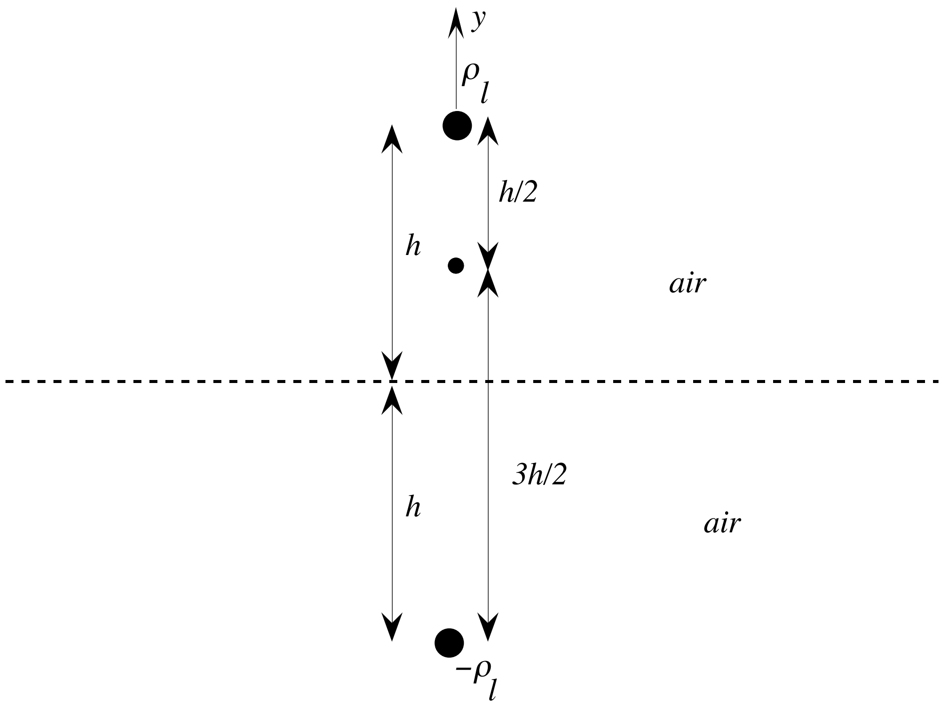
$$E^- = \frac{\rho_l}{2\pi\varepsilon_0 r} = \frac{\rho_l}{2\pi\varepsilon_0 (h + y')}$$

The total field:

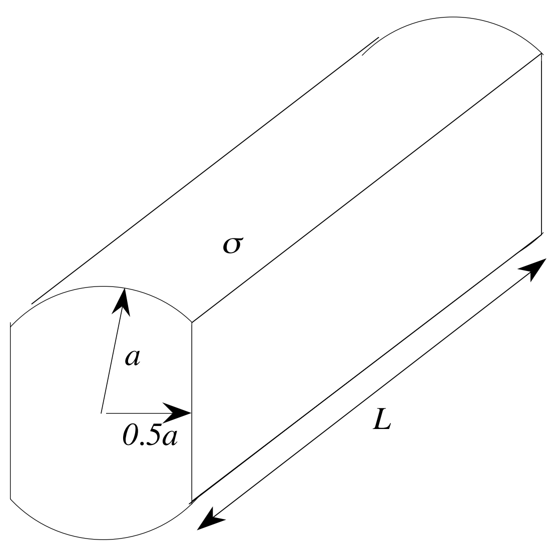
$$E = E^+ + E^- = \frac{\rho_l}{2\pi\varepsilon_0 (h - y')} + \frac{\rho_l}{2\pi\varepsilon_0 (h + y')} = \frac{\rho_l}{2\pi\varepsilon_0}\left[\frac{1}{h - y'} + \frac{1}{h + y'}\right] = \frac{\rho_l h}{\pi\varepsilon_0 \left(h^2 - y'^2\right)}$$

The potential at y=h/2 is:

$$V(y = h/2) = -\int_{y'=0}^{h/2} \frac{\rho_l h}{\pi\varepsilon_0\left(h^2 - y'^2\right)} dy'$$



e7.38. In chapter7.problems.extra. 2. A cylindrical conductor has conductivity σ, radius a and length L. Now a slice is cut on two opposite sides of the conductor to form two flat surfaces running the length of the conductor. Calculate the resistance between these two flat surfaces (i.e. as if you were measuring the resistance with an ohm-meter between the two flat surfaces). Note, the distance between the two flat surfaces is a while the curved section has radius a.



Solution: define an element of resistance as shown below. This element has length dx width h and depth L. Thus:

$$dR = \frac{dx'}{\sigma S} = \frac{dx'}{\sigma L h}$$

h is calculated as:

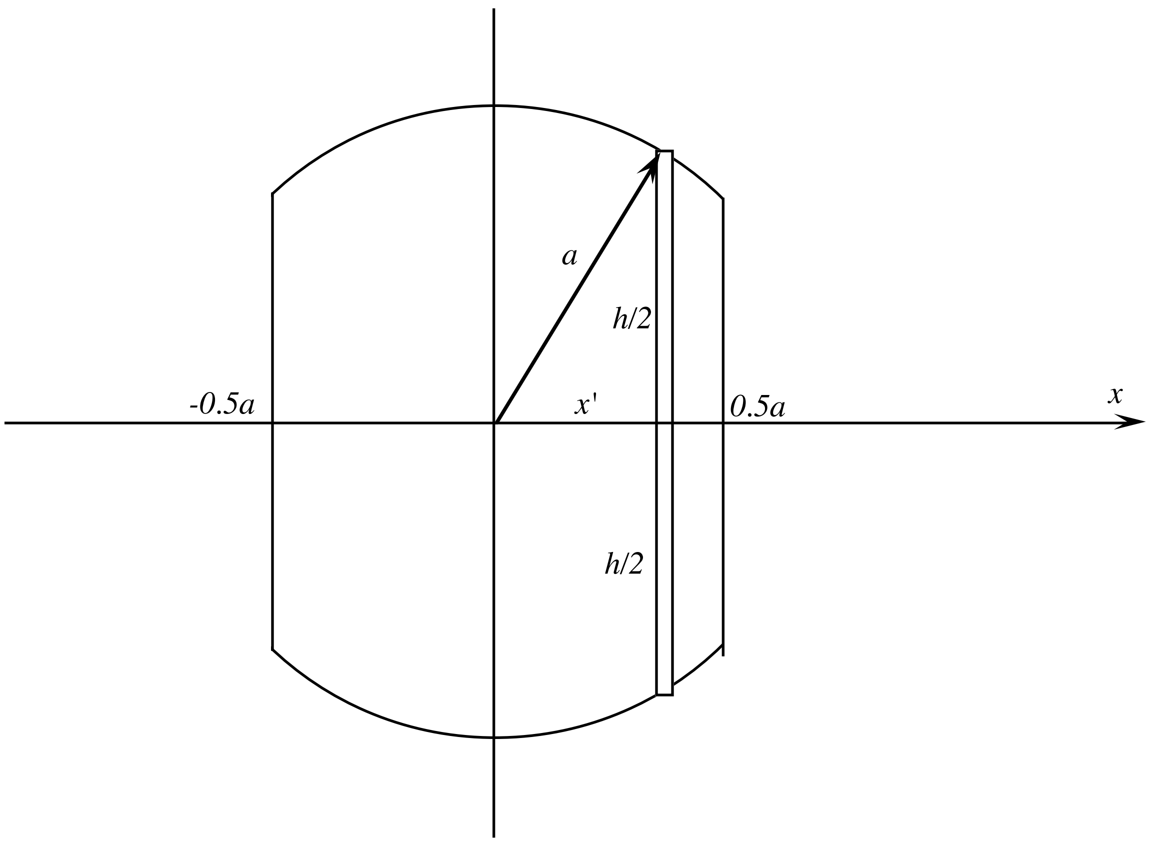
$$h = 2\sqrt{a^2 - x'^2}$$

Thus:

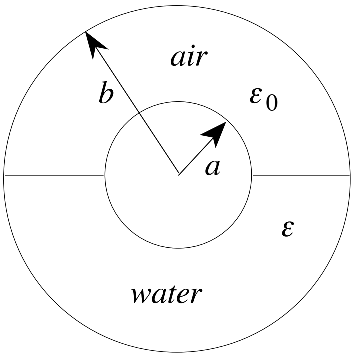
$$dR = \frac{dx,}{2\sigma L\sqrt{a^2 - x'^2}}$$

This is now inegrated across the width of the bar:

$$R = 2\int_{x'=0}^{x'=0.5a} \frac{dx'}{2\sigma L\sqrt{a^2 - x'^2}} = \frac{1}{\sigma L}\int_{x'=0}^{x'=0.5a} \frac{dx'}{\sqrt{a^2 - x'^2}} = \frac{1}{\sigma L}\left[sin^{-1}\frac{x'}{a}\right]_0^{0.5a}$$
$$= \frac{1}{\sigma L}\left[sin^{-1}\frac{0.5a}{a}\right] = \frac{1}{\sigma L}\left[sin^{-1}0.5\right] = \frac{\pi}{6\sigma L} \qquad [\Omega]$$



e4.137. In chapter4.problems.extra. 3. Two very thin spherical shells, are arranged concentrically. The inner shell has radius a, the outer, radius b. The permittivity between the shells is ε_0. Now half the space between the shells is filled with water which has permittivity ε. If exactly half the space is filled, calculate the capacitance of the device (i.e. the capacitance between the two shells).



Solution: Because the field in the spherical capacitor is exactly radial, the fact the half the capacitor is filled with water, does not change the radial nature of the field. Thus, we can calculate the capacitance of an empty capacitor and divide it by two and then do the same for a spherical capacitor filled with the dielectric. The sum of the two capacitors gives the total capacitance:

The capacitance of a spherical capacitor is calculated as follows: Using Gauss's law and assuming a total charge Q on the inner shell, the electric field intensity is:

$$E = \frac{Q}{4\pi\varepsilon_0 R^2}, \qquad a<R<b \qquad or: \qquad \mathbf{E} = \hat{\mathbf{R}}\frac{Q}{4\pi\varepsilon_0 R^2}$$

Integrating from a to b, we get the potential difference produced by the charge:

$$V_{ab} = \int_{R=b}^{a} \mathbf{E}.d\mathbf{R} = \left[\frac{Q}{4\pi\varepsilon_0 R}\right]_b^a = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Capacitance of the empty capacitor is:

$$C_{empty} = \frac{Q}{V_{ab}} = \frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}}$$

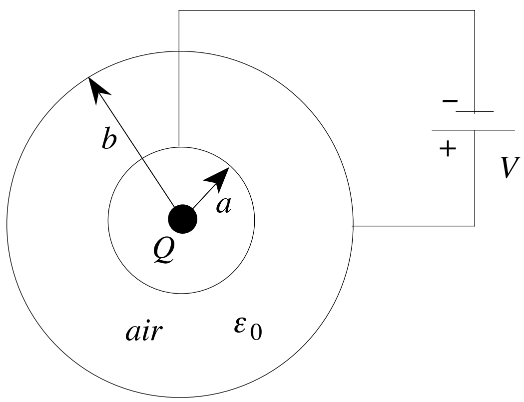
The capacitance when the whole capacitor is filled is:

$$C_{full} = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

The combined capacitance is:

$$C_t = \frac{1}{2}\frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}} + \frac{1}{2}\frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}} = \frac{2\pi}{\frac{1}{a} - \frac{1}{b}}(\varepsilon + \varepsilon_0)$$

e4.138. In chapter4.problems.extra. 4. Two very thin spherical shells, are arranged concentrically. The inner shell has radius a, the outer, radius b. The permittivity between the shells is ε_0. A potential difference V is connected across the two shells with the polarity shown. A point charge Q is placed at the center of the shells. Calculate the energy stored in the space between the two shells.



Solution: To calculate the energy it is best to separate the problem into two separate problems. One is a pherical capacitor connected to a potential V. The second is the energy due to the electric field intensity bewteen the two conducting shells produced by the point charge.

1. Due to the spherical capacitor: The capacitance is:

$$C = \frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi\varepsilon_0 ab}{b - a}$$

The energy due to capacitance is:

$$W_c = \frac{CV^2}{2} = \frac{2\pi\varepsilon_0 abV^2}{b - a} \qquad [\mathrm{J}]$$

2. Due to the field of the charge: the electric field intensity is:

$$E = \frac{Q}{4\pi\varepsilon_0 R^2}, \qquad a<R<b \qquad or: \qquad \mathbf{E} = \hat{\mathbf{R}}\frac{Q}{4\pi\varepsilon_0 R^2}$$

The energy density is:

$$w_e = \frac{\varepsilon_0 E^2}{2} = \frac{Q^2}{8\pi^2\varepsilon_0 R^4} \qquad \left[\frac{\mathrm{J}}{\mathrm{m}^3}\right]$$

This must be integrated over the volume between the two shells. Taking an element of thickness dR, gives an element of volume $4\pi R^2 dR$
This gives the energy:

$$w_q = \int_{R=a}^{R=b} \frac{\varepsilon_0 E^2}{2}\,dv = \int_{R=a}^{R=b} \frac{Q^2}{8\pi^2\varepsilon_0 R^4}\,4\pi R^2 dR = \int_{R=a}^{R=b} \frac{Q^2}{2\pi\varepsilon_0 R^2}\,dR = -\left[\frac{Q^2}{2\pi\varepsilon_0 R}\right]_a^b = \frac{Q^2}{2\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \qquad [\mathrm{J}]$$

The total energy between a and b is the sum of these two energies:

$$W = W_c + W_q = \frac{2\pi\varepsilon_0 abV^2}{b - a} + \frac{Q^2}{2\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \qquad [\mathrm{J}]$$