Electromagnetics I
Exam No. 2
November 7, 2003
Solution

Please read the exam carefully. Solve the following 4 problems. Each problem is 1/4 of the grade. To receive full credit, you must show all work. If you need to assume anything, state your assumptions clearly. Reasonable assumptions that are necessary to solve the problem will be accepted. In all problems assume properties of free space (ε_0=8.85×10^{-12} F/m, μ_0=4π×10^{-7} H/m). You can write on both sides of the page. If you need additional space ASK for additional paper and make sure you write your name on it.

Level of difficulty: 3 (most difficult), 4, 1, 2 (easiest)

e7.36. In chapter7.problems.extra. 1. A solid conductor of radius *b* and length *L* has a known conductivity σ. Two holes are now drilled into the conductor, parallel to the axis of the conductor, each hole being of radius *a* (*a*<*b*/2) (see **Figure A**). Calculate the change in resistance of the device due to the drilling of the holes. The resistance is calculated along the cylinder (i.e. as if the current flows along the cylinder).

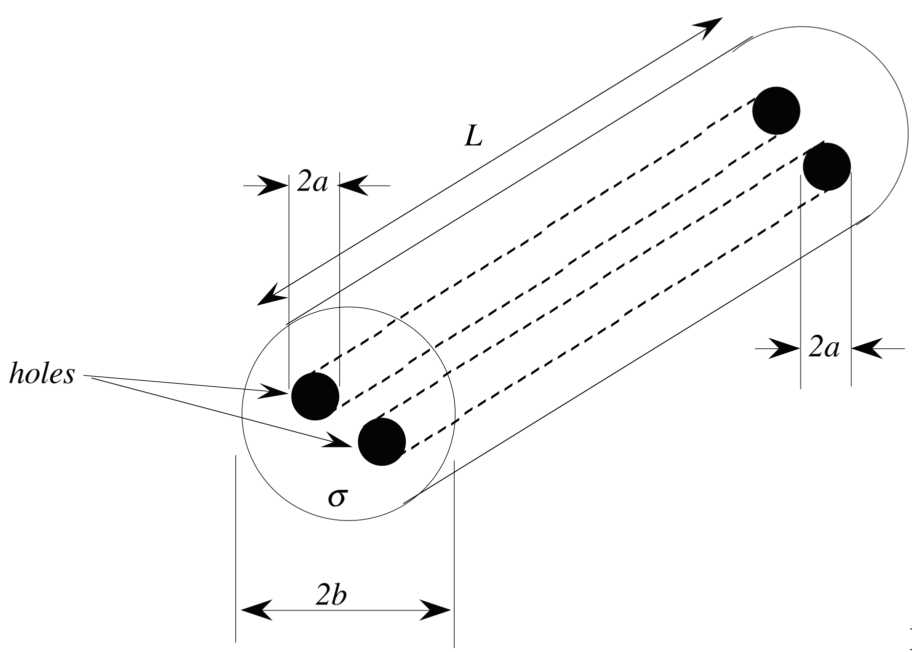

Figure A

Solution: Viewing each drilled hole as the removal of a conducting cylinder, we can still use the resistance as follows:

$$R = \frac{l}{\sigma S}$$

where S is the cross sectional area of the conductor and l is the length of the conductor. Since the cross-sectional area remains constant after drilling, the current density is constant and we have:

$$R = \frac{L}{\sigma\left(\pi b^2 - 2\pi a^2\right)}$$

Note: you can check that this is correct by adding, in parallel to this resistance the two resistances of the conducting cylinders removed by drilling. Doing so will produce the resistance of the original cylinder without the drilled holes.

The resistance without the holes is:

$$R_0 = \frac{L}{\sigma S} = \frac{L}{\sigma \pi b^2}$$

The change in resistance is the final resistance R minus the initial resistance R_0:

$$\Delta R = R - R_0 = \frac{L}{\sigma\left(\pi b^2 - 2\pi a^2\right)} - \frac{L}{\sigma \pi b^2} = \frac{L}{\sigma\pi}\left(\frac{1}{b^2 - 2a^2} - \frac{1}{b^2}\right) = \frac{L}{\sigma\pi}\left(\frac{b^2 + 2a^2 - b^2}{\left(b^2 - 2a^2\right)b^2}\right) = \frac{2a^2 L}{\sigma\pi\left(b^2 - 2a^2\right)b^2}$$

Note that this is positive indicating that the resistance increased due to the two holes.

e5.28. In chapter5.problems.extra. 2. A conducting sphere of radius a is held at a constant potential V_0 with respect to infinity. At a distance a from its surface ($2a$ from its center) the potential is $V_0/2$. Calculate the potential at a distance $a/2$ from its surface ($1.5a$ from its center).

Solution: Because the conducting sphere is at constant potential, the potential at a distance $2a$ from its center will be constant on a spherical surface of radius 2a. Thus, we have two concentric spherical surfaces, one of radius a at potential V_0 and the second at potential $V_0/2$ and radius $2a$. Thus we can use Laplace's equation in spherical coordinates (variation in potential is in the R direction only):

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) = 0 \qquad \rightarrow \qquad \frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) = 0$$

Integrating once:

$$R^2\frac{\partial V}{\partial R} = C_1 \qquad \rightarrow \qquad \frac{\partial V}{\partial R} = \frac{C_1}{R^2}$$

Integrating again:

$$V(R) = -\frac{C_1}{R} + C_2$$

Evaluating for the potential at $R=a$ and $R=2a$:

$$V(R{=}a) = -\frac{C_1}{a} + C_2 = V_0$$
$$V(R{=}2a) = -\frac{C_1}{2a} + C_2 = \frac{V_0}{2}$$

Subtracting the second from the first:

$$-\frac{C_1}{2a} = \frac{V_0}{2} \qquad \rightarrow \qquad C_1 = -V_0 a$$

From the first boundary condition:

$$-\frac{C_1}{a} + C_2 = V_0 \qquad \rightarrow \qquad V_0 + C_2 = V_0 \qquad \rightarrow \qquad C_2 = 0$$

Thus:

$$V(R) = \frac{V_0 a}{R}$$

At $R=3a/2$:

$$V(R=3a/2) = \frac{2V_0 a}{3a} = \frac{2V_0}{3}$$

e8.81. In chapter8.problems.extra. 3. A square wire frame of size $2a \times 2a$ is made of four very thin wires, carries a current I and lies in the xy-plane in free space. The frame is centered at the origin. Point P is given, at a height h above the center of the frame. Find the magnetic flux density at P (magnitude and direction) produced by the frame.

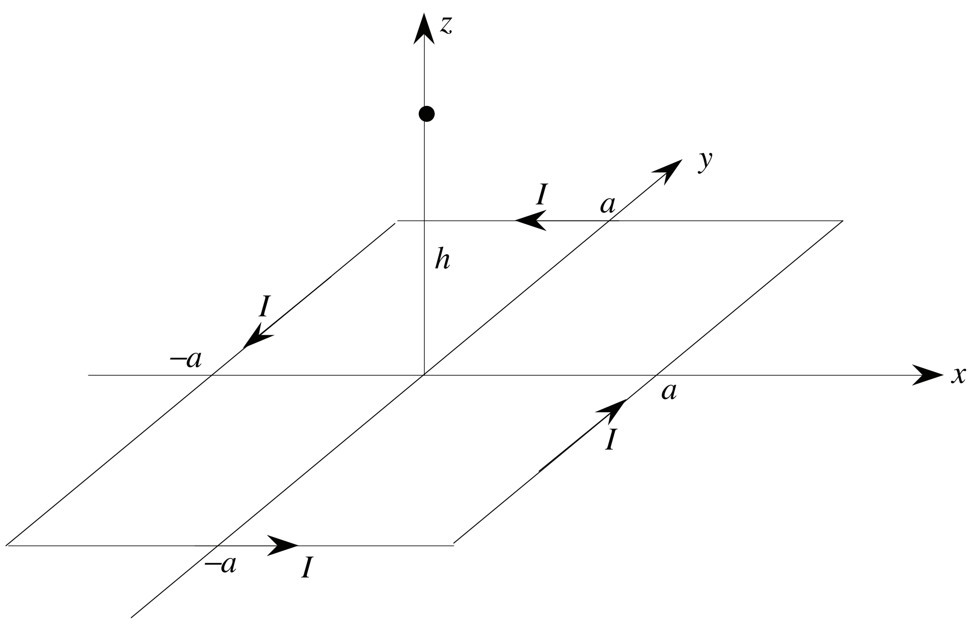

Figure A.

Solution: Each one of the four segments produces a magnetic flux density as shown in **Figure B**. We also note that each opposite pair of segments produce fields with opposing horizontal components. Thus we only need to calculate the vertical component of one segment and then multiply this by 4 to obtain the total flux density at P. To calculate the magnetic flux density due to the right segment, we use the Biot-Savart law as shown in **Figure C**. Taking the element of length in the direction of the current:

$$d\mathbf{l}' = \hat{\mathbf{y}}dy'$$

To evaluate the vector **R**, we note that $d\mathbf{l}'$ is placed at a point $(a,y'0)$ and P is at $(0,0,h)$. Thus:

$$\mathbf{R} = \hat{\mathbf{x}}(0 - a) + \hat{\mathbf{y}}(0 - y') + \hat{\mathbf{z}}(h - 0) = -\hat{\mathbf{x}}a - \hat{\mathbf{y}}y' + \hat{\mathbf{z}}h$$

Now, from the Biot-Savart law we have:

$$\mathbf{B}_{right} = \frac{\mu_0 I}{4\pi}\int_{y'=-a}^{y'=a} \frac{(\hat{\mathbf{y}}dy')\times\left(-\hat{\mathbf{x}}a - \hat{\mathbf{y}}y' + \hat{\mathbf{z}}h\right)}{\left(y'^2 + a^2 + h^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi}\int_{y'=-a}^{y'=a} \frac{\left(\hat{\mathbf{z}}a + \hat{\mathbf{x}}h\right)dy'}{\left(y'^2 + a^2 + h^2\right)^{3/2}}$$

$$= \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\int_{y'=-a}^{y'=a} \frac{ady'}{\left(y'^2 + a^2 + h^2\right)^{3/2}} + \hat{\mathbf{x}}\frac{\mu_0 I}{4\pi}\int_{y'=-a}^{y'=a} \frac{hdy'}{\left(y'^2 + a^2 + h^2\right)^{3/2}}$$

As mentioned before, the x component must cancel and we are left with:

$$\mathbf{B}_{z\text{-}right} = \hat{\mathbf{z}}\frac{\mu_0 aI}{4\pi}\int_{y'=-a}^{y'=a} \frac{dy'}{\left(y'^2 + a^2 + h^2\right)^{3/2}} = \hat{\mathbf{z}}\frac{\mu_0 aI}{4\pi}\left[\frac{y'}{\left(a^2 + h^2\right)\sqrt{y'^2 + a^2 + h^2}}\right]_{y'=-a}^{y'=a}$$

$$= \hat{\mathbf{z}}\frac{\mu_0 aI}{4\pi}\left[\frac{a}{\left(a^2 + h^2\right)\sqrt{a^2 + a^2 + h^2}} + \frac{a}{\left(a^2 + h^2\right)\sqrt{a^2 + a^2 + h^2}}\right] = \hat{\mathbf{z}}\frac{\mu_0 aI}{4\pi}\left[\frac{2a}{\left(a^2 + h^2\right)\sqrt{a^2 + a^2 + h^2}}\right]$$

Multiplying this by four we have the total field at P:

$$\mathbf{B}_P = \hat{\mathbf{z}}\frac{2\mu_0 a^2 I}{\pi\left(a^2 + h^2\right)\sqrt{a^2 + a^2 + h^2}} \qquad [\mathrm{T}]$$

Note: it was not necessary to perform the integration. The following answer is also appropriate:

$$\mathbf{B} = \hat{\mathbf{z}}4\frac{\mu_0 aI}{4\pi}\int_{y'=-a}^{y'=a} \frac{dy'}{\left(y'^2 + a^2 + h^2\right)^{3/2}} \qquad [\mathrm{T}]$$

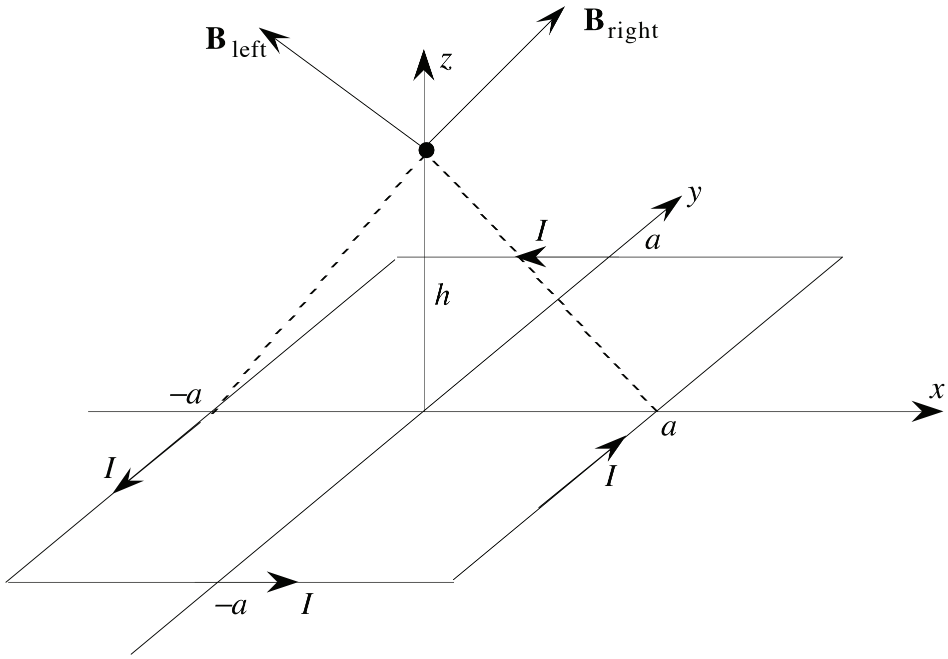

Figure B

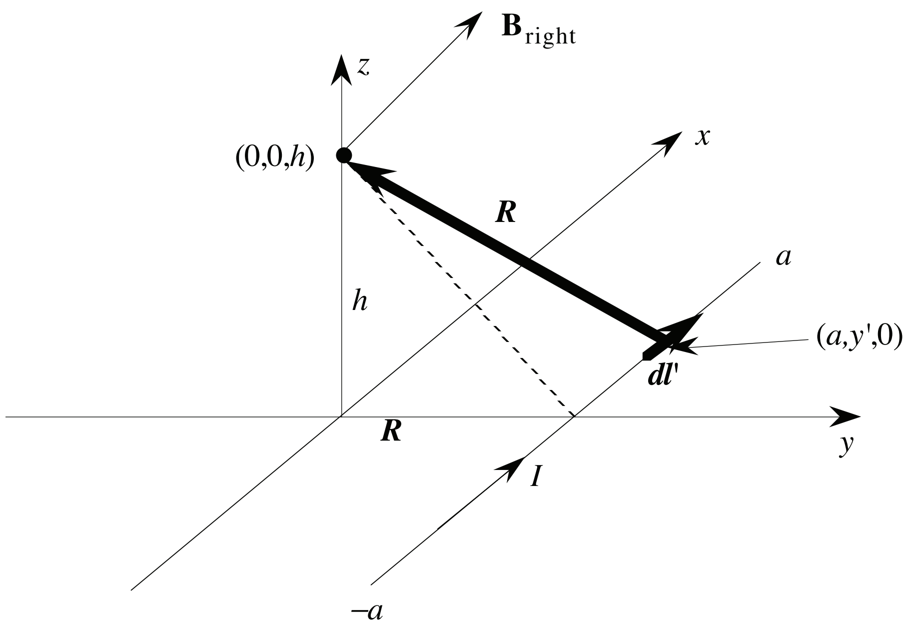

Figure C

e5.29. In chapter5.problems.extra. 4. Two point charges are placed in free space at a distance d from each other in free space. One charge is positive and equal to Q, the second is negative and equal to $-Q$ as shown in **Figure A**. Point P is the center point between the two charges. Now a conductor is brought into the vicinity of the two charges so that the configuration is as in **Figure B.**

a. Find the change in the electric field intensity at point P caused by the conductor.

b. Calculate the change in the potential at point P caused by the conductor.

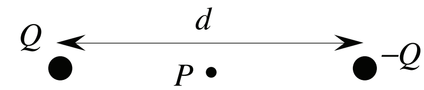

Figure A

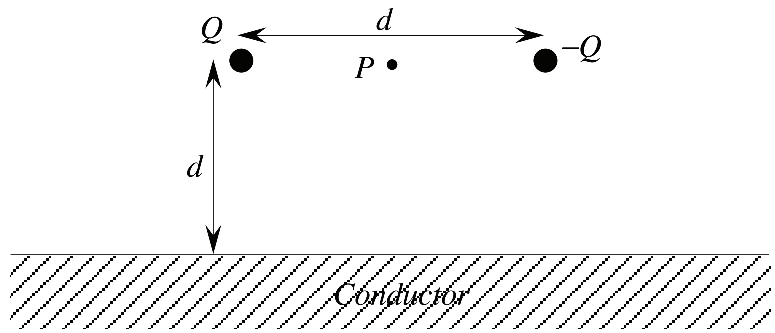

Figure B

Solution: Calculate the potential in **Figures A** and **B** and take the difference.

a. The electric field intensity at P in **Figure A** is:

$$\mathbf{E}_p = \hat{\mathbf{x}} 2 \frac{Q}{4\pi\varepsilon_0 (d/2)^2} = \hat{\mathbf{x}} \frac{2Q}{\pi\varepsilon_0 d^2}$$

where we assume the positive x-axis points to the right. In **Figure B,** we have an image system as shown in **Figure C** where the directions of the fields are also shown. These two fields have equal magnitudes and therefore equal and opposing y components. The x components add up as follows:

$$\mathbf{E}_x^+ = \mathbf{E}^+ \cos\alpha = -\hat{\mathbf{x}} \frac{Q}{4\pi\varepsilon 0\left[(2d)^2 + (d/2)^2\right]} \frac{(d/2)}{\left[(2d)^2 + (d/2)^2\right]^{1/2}} = -\hat{\mathbf{x}} \frac{Qd}{8\pi\varepsilon 0\left[(2d)^2 + (d/2)^2\right]^{3/2}} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

Since $\mathbf{E}^-$ produces an identical component, the total field due to the two image charges is twice this much. To this we must add the field due to the two original charges but, since we are only interested in the change in the electric field, we then subtract the original field and get:

$$\Delta\mathbf{E} = -\hat{\mathbf{x}} \frac{Qd}{4\pi\varepsilon 0\left[(2d)^2 + (d/2)^2\right]^{3/2}} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

b. Zero. The potential between two opposite charges is zero. The presence of the conductor does not change this condition.

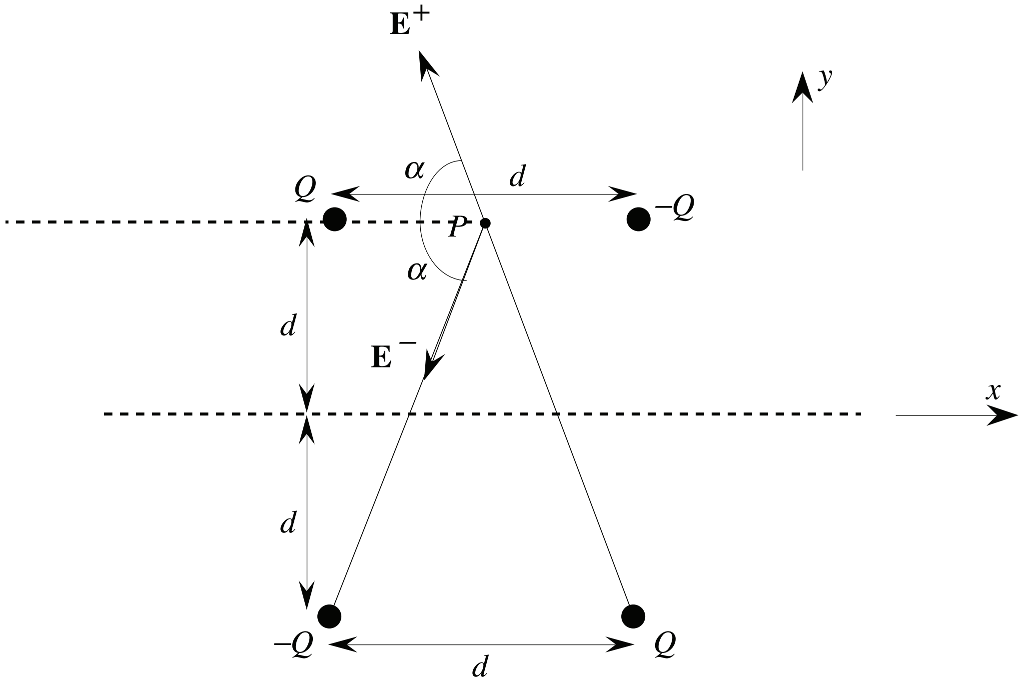

Figure C