Electromagnetics I
Exam No. 2
November 1, 2002

Please read the exam carefully. Solve the following 4 problems. Each problem is 1/4 of the grade. To receive full credit, you must show all work. If you need to assume anything, state your assumptions clearly. Reasonable assumptions that are necessary to solve the problem will be accepted. In all problems assume properties of free space (ε_0=8.85×10^{-12} F/m, μ_0=4π×10^{-7} H/m). You can write on both sides of the page. If you need additional space ASK for additional paper and make sure you write your name on it.

Level of difficulty: 4 (most difficult), 1, 3, 2 (easiest)

1. An aluminum bar is used as a lightning condit (connecting a lightning bar to ground) in a building and is made as in **Figure A**. The conductivity of aluminum is 3.6×10^7 [S/m]. Since it is found that the bar contains too much material and therefore is too expensive, it is proposed to cut a hole in it to reduce the amount of material as shown in **Figure B**. Calculate the change in resistance due to this change.

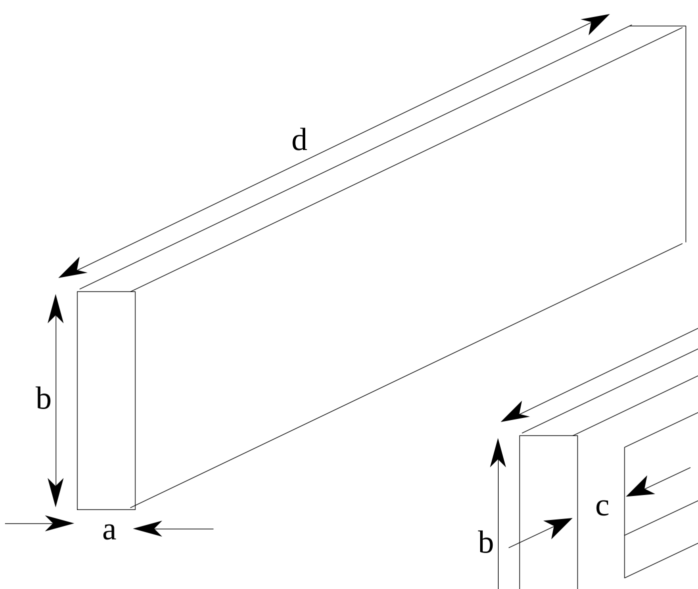


Figure A

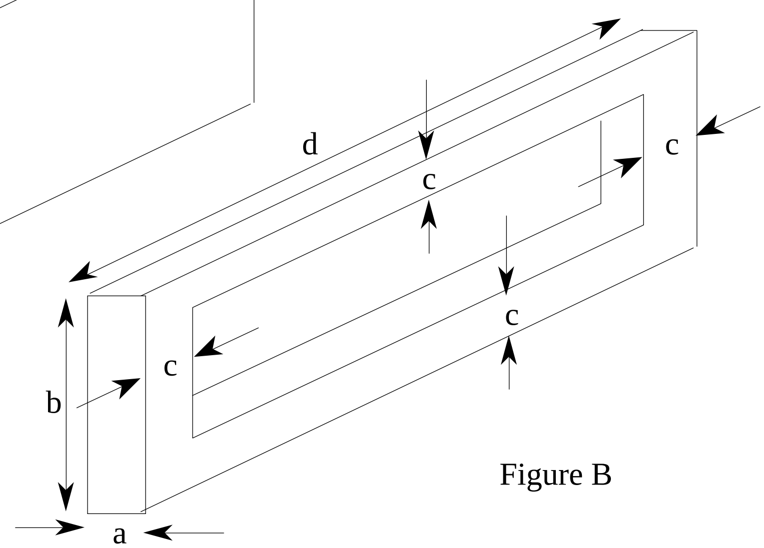


Figure B

a=10mm, b=200mm, c=10mm, d=10m

Solution: The resistance of the solid conductor is calculated directly as:

$$R_s = \frac{L}{\sigma S} = \frac{d}{\sigma ab}$$

For the hollow conductor, we view this as being made of four sections as shown in **Figure C**. Each of the two end sections has resistance:

$$R_e = \frac{L}{\sigma S} = \frac{c}{\sigma ab}$$

The two horizontal sections, each has resistance:

$$R_h = \frac{L}{\sigma S} = \frac{d - 2c}{\sigma ac}$$

The fouir resistances are connected as shown in **Figure D**. The total resistance is:

$$R_t = 2R_e + R_h/2 = \frac{2c}{\sigma ab} + \frac{d - 2c}{2\sigma ac}$$

The change in resistance is therefore:

$$\Delta R = R_t - R_s = \frac{2c}{\sigma ab} + \frac{d - 2c}{2\sigma ac} - \frac{d}{\sigma ab}$$

$$\Delta R = \frac{1}{\sigma}\left(\frac{2c}{ab} + \frac{d - 2c}{2ac} - \frac{d}{ab}\right) = \frac{1}{3.6\times 10^7}\left(\frac{2\times 0.01}{0.01\times 0.2} + \frac{10 - 2\times 0.01}{2\times 0.01\times 0.01} - \frac{10}{0.01\times 0.2}\right)$$

$$= \frac{10^{-7}}{3.6}(10 + 50000 - 100 - 5000) = 0.0012475 \qquad [\Omega]$$

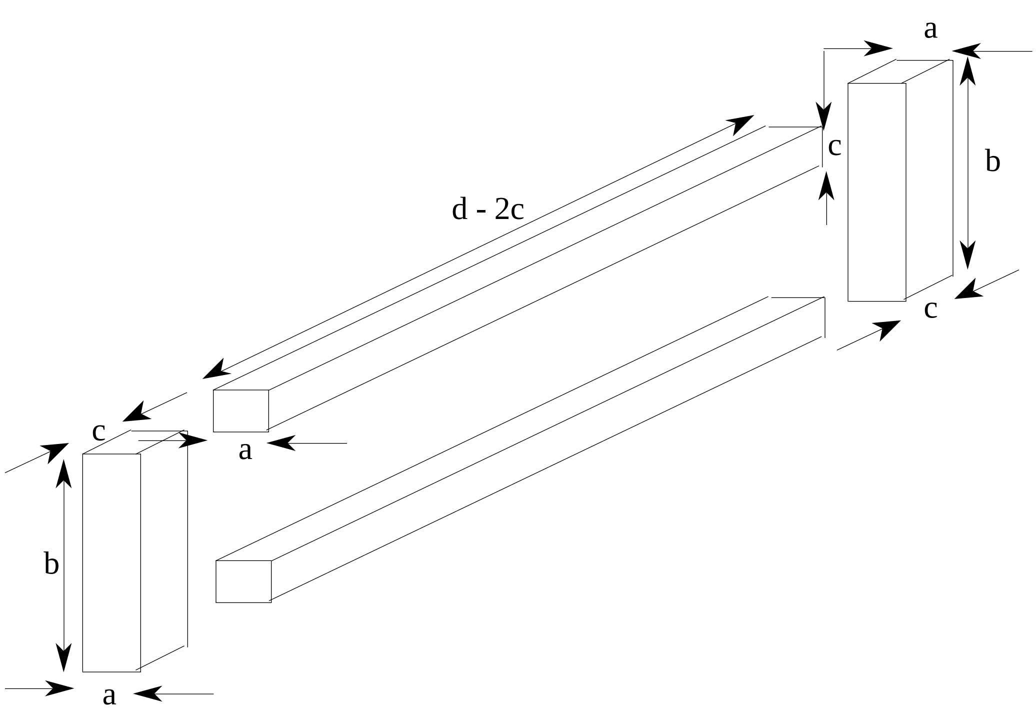


Figure C

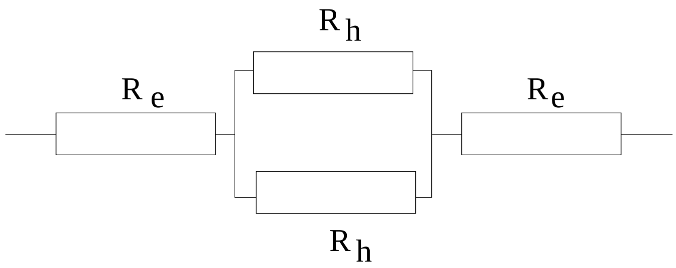
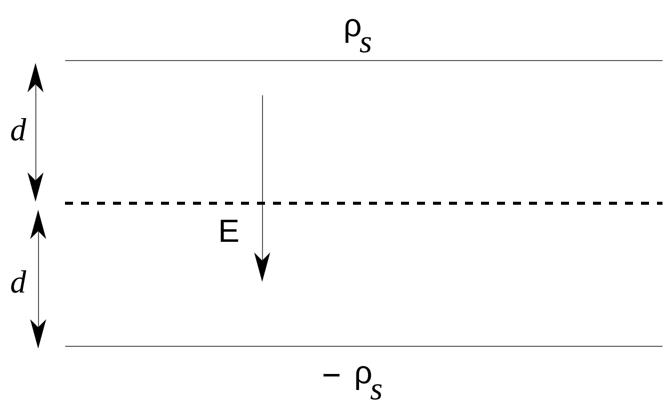


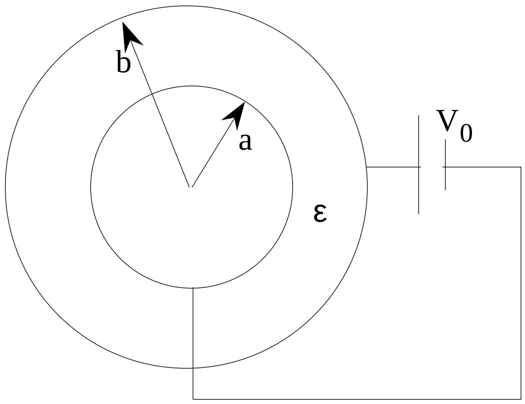
Figure D

2. A plane of charge with charge density ρ_s [C/m^2] is placed at a distance *d* above a conducting ground, parallel to the ground. Calculate the electric field intensity above and below the plane of charge.

Solution: The conducting surface acts as a mirror and therefore the equivalent configuration is that of two sheets, one charged with a positive charge density (the original sheet) and one charged with negative charge density at a distance 2*d* from each other (see figure). The field intensity between the two sheets id ρ_s/ε_0. Thus:
The field is zero above the charged plane.
The field is ρ_s/ε_0 between the plane and the grounded conductor.
The field is zero inside the conducting ground.



3. A spherical configuration is made of a thin inner shell of radius *a*=0.2m and a thin outer shell of radius *b*=0.201m. The material between the shells has permittivity of $4\varepsilon_0$. A potential difference V_0=120V is connected across the two shells as shown. Calculate the energy required to remove the dielectric material from between the shells assuming nothing else changes.



Solution: The easiest way is to calculate the capacitance with the dielectric and the capacitance without the dielectric and calculate the energy from capacitance in each case. The difference between the final energy (without dielectric) and the initial energy (with dielectric) is the energy required to remove the dielectric.

The capacitance of a spherical capacitor of inner radius *a* and outer radius *b* is:

$$C_\varepsilon = \frac{16\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}}, \qquad C_{\varepsilon_0} = \frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}}$$

The energy in a capacitor is:

$$W = C\frac{V^2}{2}$$

Thus:

Final energy is due to capacitor without the dielectric:

$$W_f = C_{\varepsilon_0}\frac{V^2}{2} = \frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}}\frac{V^2}{2} = \frac{2\pi\varepsilon_0 abV^2}{b - a}$$

The initial energy is:

$$W_i = C_\varepsilon\frac{V^2}{2} = \frac{16\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}}\frac{V^2}{2} = \frac{8\pi\varepsilon_0 abV^2}{b - a}$$

Thus:

$$\Delta W = W_f - W_i = \frac{2\pi\varepsilon_0 abV^2}{b - a} - \frac{8\pi\varepsilon_0 abV^2}{b - a} = -\frac{6\pi\varepsilon_0 abV^2}{b - a} \qquad [\mathrm{J}]$$

$$\Delta W = -\frac{6\pi\varepsilon_0 abV^2}{b - a} = -\frac{6\times\pi\times 8.854\times 10^{-12}\times 0.2\times 0.201\times 120^2}{0.001} = -9.96\times 10^{-5} \qquad [\mathrm{J}]$$

The negative change in energy indicates that the initial energy is higher meaning that in the process, the potential energy of the sphere has diminished.

4. Overhead power lines are made of a steel core (used mostly for strength) and an outer coating of aluminum. Consider a 100 km DC power line as shown made of a single overhead wire and a ground return, supplying power to a 100 Ω load as shown in **Figure A**. The dimensions of the conductors cross section is shown in **Figure B**. Calculate the magnetic field intensity inside the steel and aluminum section of the conductor neglecting the influence of the ground.

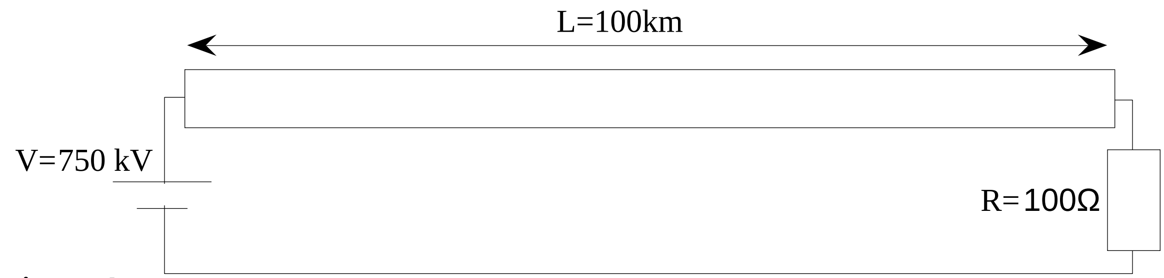
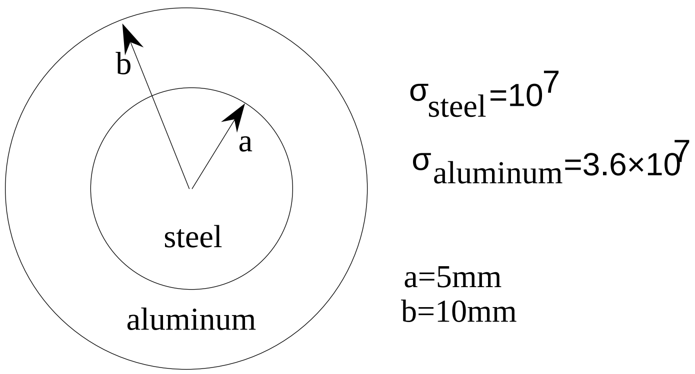


Figure A.



Cross-section of power line

Figure B.

Solution: To find the magnetic field intensity in the steel and aluminun sections, we first need the current or the current.
The resistance of the inner steel conductor is:

$$R_s = \frac{L}{\sigma S} = \frac{L}{\sigma_s \pi a^2} \qquad [\Omega]$$

The resistance of the outer aluminum conductor is:

$$R_{al} = \frac{L}{\sigma S} = \frac{L}{\sigma_{al}\pi(b^2 - a^2)} \qquad [\Omega]$$

These two resistances are connected in parallel so we could calculate the total current. Instead, we calculate the current and the current density in each of the two materials:

$$I_{al} = \frac{V}{R_L + R_{al}} = \frac{V}{R_L + \dfrac{L}{\sigma_{al}\pi(b^2 - a^2)}} = \frac{V\sigma_{al}\pi(b^2 - a^2)}{\sigma_{al}\pi(b^2 - a^2)R_L + L} \qquad [\text{A}]$$

By dividing this by the cross-sectional area we get the current density in aluminum:

$$J_{al} = \frac{I_{al}}{\pi(b^2 - a^2)} = \frac{V\sigma_{al}}{\sigma_{al}\pi(b^2 - a^2)R_L + L} \qquad \left[\frac{\text{A}}{\text{m}^2}\right]$$

Similarly for the steel conductor:

$$I_s = \frac{V\sigma_s\pi a^2}{\sigma_{al}\pi a^2 R_L + L} \qquad [\text{A}]$$

$$J_s = \frac{I_s}{\pi(b^2 - a^2)} = \frac{V\sigma_s}{\sigma_s\pi a^2 R_L + L} \qquad \left[\frac{\text{A}}{\text{m}^2}\right]$$

Now that we have the current (or current density), the magnetic field intensity is found using Ampere's law. In steel, at a distance $0<r<a$, the magnetic field intensity is:

$$\int_C \mathbf{H}.d\mathbf{l} = I_{enc.} \qquad \rightarrow \qquad 2\pi rH = J_s\pi r^2$$

or:

$$H_s = \frac{J_s\pi r^2}{2\pi r} = \frac{V\sigma_s\pi r^2}{2\pi r\left(\sigma_s\pi a^2 R_L + L\right)} = \frac{V\sigma_s r}{2\left(\sigma_s\pi a^2 R_L + L\right)} \qquad \left[\frac{\text{A}}{\text{m}}\right]$$

The same applies in aluminum. At a distance $a<r<b$, the total current enclosed contains all the current in the steel conductor plus that current in the aluminum up to that point. Thus:

$$H_{al} = \frac{J_s\pi a^2 + J_{al}\pi\left(r^2 - a^2\right)}{2\pi r} = \frac{V\sigma_s\pi a^2}{2\pi r\left(\sigma_s\pi a^2 R_L + L\right)} + \frac{V\sigma_{al}\pi\left(r^2 - a^2\right)}{2\pi r\left[\sigma_{al}\pi(b^2 - a^2)R_L + L\right]} =$$
$$\frac{V\sigma_s a^2}{2r\left(\sigma_s\pi a^2 R_L + L\right)} + \frac{V\sigma_{al}\left(r^2 - a^2\right)}{2r\left[\sigma_{al}\pi(b^2 - a^2)R_L + L\right]} \qquad \left[\frac{\text{A}}{\text{m}}\right]$$

Numerically these give (these calculations are not necessary):

$$H_s = \frac{750000\times 1\times 10^7 r}{2\left(10^7\times\pi\times 0.005^2\times 100 + 100000\right)} = 2.1\times 10^7 r \qquad \left[\frac{\text{A}}{\text{m}}\right]$$

$$H_{al} = \frac{750000\times 1\times 10^7\times 0.005^2}{2\times r\left(1\times 10^7\times\pi\times 0.005^2 100 + 100000\right)} + \frac{750000\times 3.6\times 10^7\left(r^2 - 0.005^2\right)}{2\times r\left[3.6\times 10^7\times\pi\times(0.01^2 - 0.001^2)100 + 100000\right]}$$
$$= \frac{525.1}{r} + 1.1\times 10^7 r - \frac{276.7}{r} \qquad \left[\frac{\text{A}}{\text{m}}\right]$$