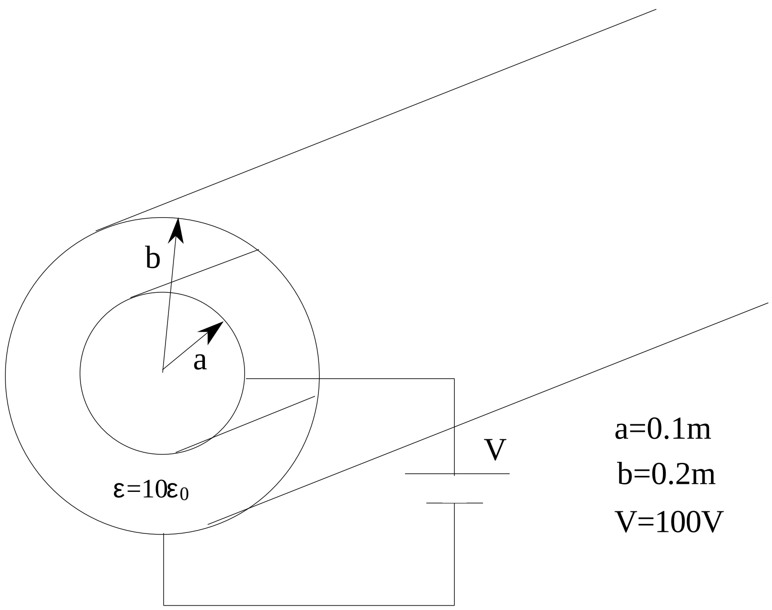
Electromagnetic Fields
Exam No. 1
September 28, 2001

Solve the following 3 problems. Each problem is 1/3 of the grade. To receive full credit, you must show all work. If you need to assume anything, state your assumptions clearly. Reasonable assumptions that are necessary to solve the problem will be accepted. In all problems assume properties of free space ($\varepsilon_0=8.85\times10^{-12}$ F/m). You can write on both sides of the page. If you need additional space ASK for additional paper and make sure you write your name on it.

1. A very long (infinite) coaxial cable is given as shown. A potential difference is connected between the inner and outer conductors.
Calculate the electric field intensity between the two conductors (magnitude and direction).



a=0.1m
b=0.2m
V=100V

Solution: Assume a surface charge density on the inner conductor. Calculate the electric field intensity in terms of the unknown, assumed charge density. Integrate between *a* and *b* to find the potential difference. Set to the given potential difference to calculate the charge density. Now substitute back into the electric field intensity to find the electric field intensity.

With an assumed charge density on the inner conductor, define a Gaussian surface in the form of a cylinder of radius *r* and length *L* such that $a<r<b$. The total charge enclosed by the Gaussian surface is $2\pi aL\rho_s$. Thus:

$$\oint_s \mathbf{E}.d\mathbf{s} = \frac{Q}{\varepsilon} \quad \rightarrow \quad E(2\pi rL) = \frac{2\pi aL\rho_s}{\varepsilon} \quad \rightarrow \quad E = \frac{a\rho_s}{\varepsilon r} \quad \rightarrow \quad \mathbf{E} = \hat{\mathbf{r}}\frac{a\rho_s}{\varepsilon r}$$

Note that the electric field intensity points away from the inner conductor since we assumed a positive surface charge density. The potential difference between *a* and *b* is:

$$V_{ab} = -\int_b^a \mathbf{E}.d\mathbf{l} = -\int_b^a \hat{\mathbf{r}}\frac{a\rho_s}{\varepsilon r}.\hat{\mathbf{r}}dr = -\frac{a\rho_s}{\varepsilon}\int_b^a \frac{1}{r}dr = -\frac{a\rho_s}{\varepsilon}\ln\frac{a}{b} = \frac{a\rho_s}{\varepsilon}\ln\frac{b}{a}$$

This potential difference equals the given potential *V*:

$$V = \frac{a\rho_s}{\varepsilon}\ln\frac{b}{a} \quad \rightarrow \quad \rho_s = \frac{\varepsilon V}{a\ln\frac{b}{a}} \quad \left[\frac{\mathrm{C}}{\mathrm{m}^2}\right]$$

Substituting into the electric field intensity we get:

$$\mathbf{E} = \hat{\mathbf{r}}\frac{a\rho_s}{\varepsilon r} = \hat{\mathbf{r}}\frac{a\varepsilon V}{\varepsilon ra\, ln\frac{b}{a}} = \hat{\mathbf{r}}\frac{V}{r\, ln(b/a)} = \hat{\mathbf{r}}\frac{100}{r\, ln(0.2/0.1)} = \hat{\mathbf{r}}\frac{100}{r\, ln(0.2/0.1)} = \hat{\mathbf{r}}\frac{144.27}{r} \qquad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

2. A conducting sphere of radius a=0.1 m is at a potential of 1000 V. A point charge equal to −1 nC (-10^{-9} C) is placed at a distance R=10m from the center of the sphere and held in place. Calculate the force that attracts the point charge to the sphere if both charge and sphere are in free space.

Solution: Method A. Calculate the electric field intensity the sphere produces at the point charge. To do so, assume a surface charge density, calculate the electric field intensity and integrate from infinity to R. Then set this potential to 1000V to find the charge density. Now the electric field intensity at R can be calculated and the force equals $\mathbf{F}=q\mathbf{E}$.
Method B: Assume a total charge Q on the sphere. Calculate the potential due to a point charge Q at a distance a and set this to be equal to 1000V. This relation gives the charge Q. Calculate the force between two point charges one q and one Q using Coulomb's law. Method A is shown here

Assuming a total charge Q, uniformly distributed on the surface of the sphere, we find from Gauss's law:

$$\mathbf{E} = \hat{\mathbf{R}}\frac{Q}{4\pi\varepsilon_0 R^2}$$

Either by integrating this from ∞ to $R=a$ or by recalling the potential of a point charge at a distance $R=a$, we find:

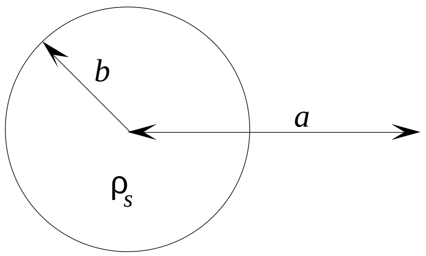
$$V = \frac{Q}{4\pi\varepsilon_0 a} \qquad \rightarrow \qquad Q = 4\pi\varepsilon_0 aV$$

The force is therefore:

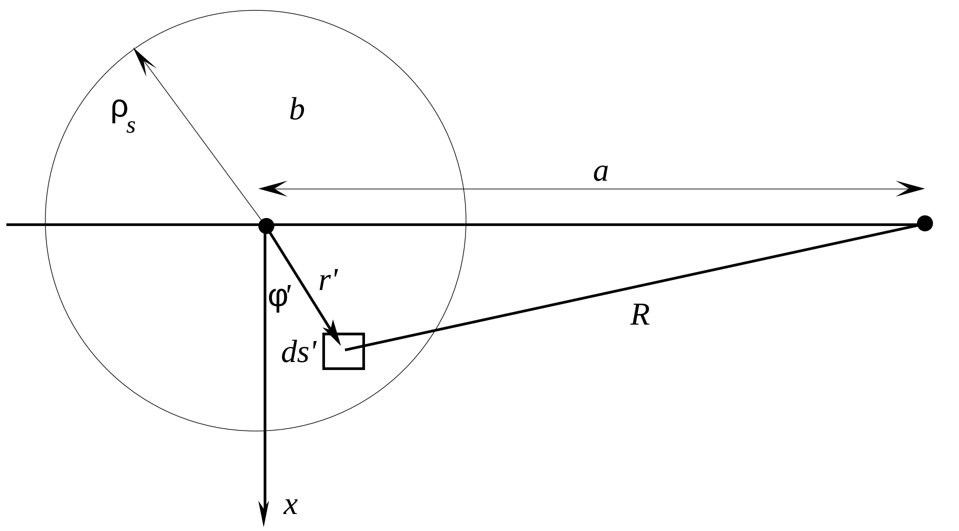
$$|\mathbf{F}| = \frac{Qq}{4\pi\varepsilon_0 R^2} = \frac{(4\pi\varepsilon_0 aV)q}{4\pi\varepsilon_0 R^2} = \frac{aVq}{R^2} = \frac{0.1\times1000\times10^{-9}}{10^2} = 10^{-9} \qquad [\mathrm{N}]$$

Note: this force points from the positive charge (sphere) to the negative charge (q) for the sphere and from the point charge to the sphere for the point charge.

3.
a. Find the potential of a very thin circular plate at a distance a from its centre, in the plane of the plate. The plate has auniform charge distribution ρ_s [C/m^2] and is located in free space.
b. Determine the electric field intensity at a distance a from the centre of the plate, in the plane of the plate.
c. Find the electric field intensity and the electric potential if a>>b and show that the problem then can be solved using Gauss' law



Solution: Define an element of area, and therefore an element of charge at a distance r' from the center of the plate as shown in the figure below.



Note that the choice of coordinates is arbitrary but one should use them judiciously. Now, the distance R may be written as follows:

$$R = \sqrt{(a - r'\sin\varphi')^2 + (r'\cos\varphi')^2} = \sqrt{a^2 + r'^2 - 2ar'\sin\varphi'}$$

The element of area is $r'd\varphi'dr'$ and the element of charge is $\rho_s r'd\varphi'dr'$.

The potential at a distance a from the center of the disk due to this element of charge is:

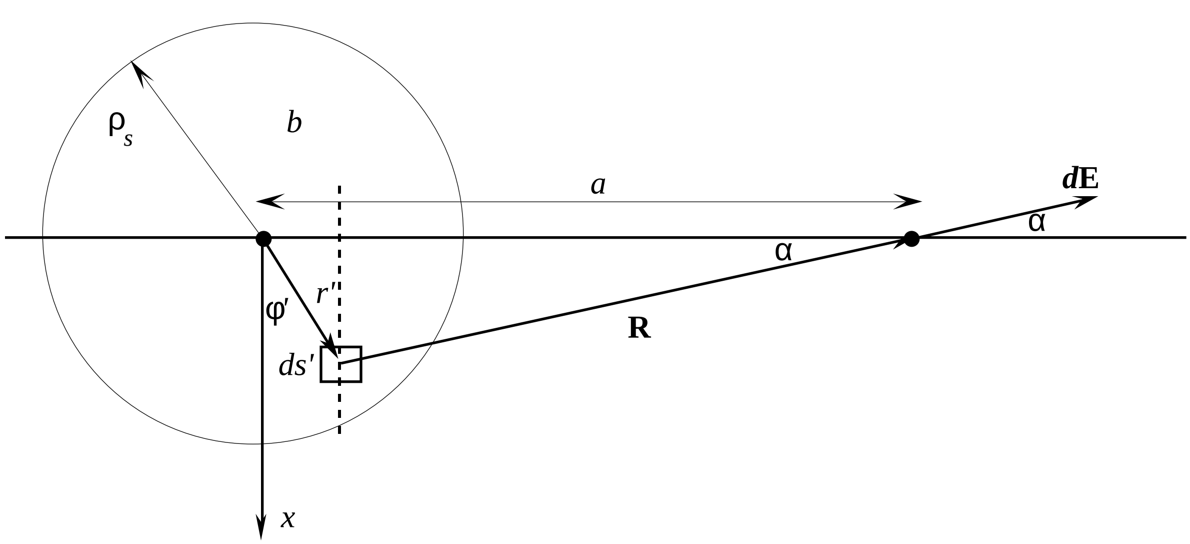
$$dV = \frac{\rho_s r'dr'd\varphi'}{4\pi\varepsilon_0\sqrt{a^2 + r'^2 - 2ar'\sin\varphi'}}$$

The potential at a distance a from the center of the disk is:

$$V = 2\int_{r'=0}^{r'=b}\int_{\varphi'=-\pi/2}^{\varphi'=\pi/2}\frac{\rho_s r'dr'd\varphi'}{4\pi\varepsilon_0\sqrt{a^2 + r'^2 - 2ar'\sin\varphi'}} = \frac{\rho_s}{2\pi\varepsilon_0}\int_{r'=0}^{r'=b}\int_{\varphi'=-\pi/2}^{\varphi'=\pi/2}\frac{r'dr'd\varphi'}{\sqrt{a^2 + r'^2 - 2ar'\sin\varphi'}}$$

Note that the limits of integration are decided based on the particular choice of coordinates.

To calculate the electric field intensity, we use the same idea but have to consider the vector notation. The element of charge defines an electric field intensity in the direction of **R** as shown below:



The length of the vector R has not changed and we can write:

$$dE = \hat{\mathbf{R}} \frac{\rho_s r' dr' d\varphi'}{4\pi\varepsilon_0 \left(a^2 + r'^2 - 2ar' \sin\varphi'\right)}$$

From symmetry considerations it is clear that the field can only have a horizontal component (the vertical component cancels due to a symmetric element of charge located above the axis). Thus, we can write:

$$dE_h = \frac{\rho_s r' dr' d\varphi' \cos\alpha}{4\pi\varepsilon_0 \left(a^2 + r'^2 - 2ar' \sin\varphi'\right)} = \frac{\rho_s (a - r' \sin\varphi') r' dr' d\varphi'}{4\pi\varepsilon_0 \left(a^2 + r'^2 - 2ar' \sin\varphi'\right)^{3/2}}$$

The electric field is therefore:

$$E_h = 2\int_{r'=0}^{r'=b} \int_{\varphi'=-\pi/2}^{\varphi'=\pi/2} \frac{\rho_s (a - r' \sin\varphi') r' dr' d\varphi'}{4\pi\varepsilon_0 \left(a^2 + r'^2 - 2ar' \sin\varphi'\right)^{3/2}} = \frac{\rho_s}{2\pi\varepsilon_0} \int_{r'=0}^{r'=b} \int_{\varphi'=-\pi/2}^{\varphi'=\pi/2} \frac{(a - r' \sin\varphi') r' dr' d\varphi'}{\left(a^2 + r'^2 - 2ar' \sin\varphi'\right)^{3/2}}$$

In the system of coordinates shown, this would be in the *y* direction.

If we assume a>>b, we can write the following for voltage:

$$V = \frac{\rho_s}{2\pi\varepsilon_0} \int_{r'=0}^{r'=b} \int_{\varphi'=-\pi/2}^{\varphi'=\pi/2} \frac{r' dr' d\varphi'}{\sqrt{a^2 + r'^2 - 2ar' \sin\varphi'}} \approx \frac{\rho_s}{2\pi\varepsilon_0} \int_{r'=0}^{r'=b} \int_{\varphi'=-\pi/2}^{\varphi'=\pi/2} r' dr' d\varphi' \left(a^2 - 2ar' \sin\varphi'\right)^{-1/2}$$

Now, taking the term in brackets we write:

$$\left(a^2 - 2ar' \sin\varphi'\right)^{-1/2} = \frac{1}{a}\left(1 - \frac{2r' \sin\varphi'}{a}\right)^{-1/2} \approx \frac{1}{a}\left(1 + \frac{2r' \sin\varphi'}{a}\right)$$

where the binomial expansion $(1 - x)^n \approx 1 - nx$ with $n=-1/2$, $x=2r'\sin\varphi'/a$ were used. Note that $x<1$ as required. With this the integral is:

$$V \approx \frac{\rho_s}{2\pi\varepsilon_0} \int_{r'=0}^{r'=b} \int_{\varphi'=-\pi/2}^{\varphi'=\pi/2} r' dr' d\varphi' \frac{1}{a}\left(1 + \frac{2r' \sin\varphi'}{a}\right) = \frac{\rho_s}{2\pi\varepsilon_0} \int_{r'=0}^{r'=b} \int_{\varphi'=-\pi/2}^{\varphi'=\pi/2} \left(\frac{r'}{a} + \frac{2r'^2 \sin\varphi'}{a}\right) dr' d\varphi'$$

Integrating first over φ':

$$V \approx \frac{\rho_s}{2\pi\varepsilon_0} \int_{r'=0}^{r'=b} \left. \left(\frac{r'\varphi'}{a} - \frac{2r'^2 \cos\varphi'}{a}\right) \right|_{-\pi/2}^{\pi/2} dr' = \frac{\rho_s}{2\pi\varepsilon_0} \int_{r'=0}^{r'=b} \frac{r'\pi}{a} = \frac{\rho_s}{2a\varepsilon_0} \int_{r'=0}^{r'=b} r' = \frac{\rho_s}{2a\varepsilon_0} \left. \frac{r'^2}{2} \right|_0^b = \frac{\rho_s b^2}{4a\varepsilon_0}$$

Note that this is the same result as would be obtained froma point charge equal to the total charge on the plate. In this case, Gauss' law applies.

For the electric field intensity:

$$\frac{(a - r' \sin\varphi') r' dr' d\varphi'}{\left(a^2 + r'^2 - 2ar' \sin\varphi'\right)^{3/2}} \approx \frac{(a - r' \sin\varphi') r' dr' d\varphi'}{\left(a^2 - 2ar' \sin\varphi'\right)^{3/2}} = (a - r' \sin\varphi') r' dr' d\varphi' \left(a^2 - 2ar' \sin\varphi'\right)^{-3/2}$$

$$= \frac{(a - r' \sin\varphi')}{a^3} r' dr' d\varphi' \left(1 - \frac{2r' \sin\varphi'}{a^2}\right)^{-3/2} = \frac{(a - r' \sin\varphi')}{a^3} r' dr' d\varphi' \left(1 + \frac{3r' \sin\varphi'}{a^2}\right)$$

Collecting terms, we get:

$$E_h \approx \frac{\rho_s}{2\pi a^3 \varepsilon_0} \int_{r'=0}^{r'=b} \int_{\varphi'=-\pi/2}^{\varphi'=\pi/2} \left(ar' + \frac{3r'^2 sin\varphi'}{a} - r'^2 sin\varphi' - \frac{3r'^3 sin\varphi'}{a} \right) dr' d\varphi'$$

To simplify integration we note that the integration on φ' on the second and third and 4th terms is zero. Thus, we have:

$$E_h \approx \frac{\rho_s}{2\pi a^3 \varepsilon_0} \int_{r'=0}^{r'=b} ar' dr' = \frac{\rho_s b^2}{4\pi a^2 \varepsilon_0}$$