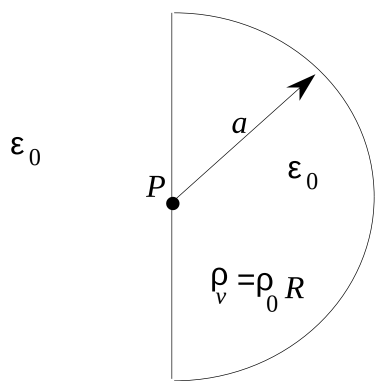
Electromagnetics I
Exam No. 1
October 4, 2002

Solve the following 4 problems. Each problem is 1/4 of the grade. To receive full credit, you must show all work. If you need to assume anything, state your assumptions clearly. Reasonable assumptions that are necessary to solve the problem will be accepted. In all problems assume properties of free space (ε_0=8.85×10^{-12} F/m). You can write on both sides of the page. If you need additional space ASK for additional paper and make sure you write your name on it.

1. A half sphere is made of a nonconducting material of permittivity ε_0. The radius of the sphere is *a* and the sphere is charged with a nonuniform charge density given as $\rho_v=\rho_0 R$ where *R* is the radial distance from the center of the sphere. The half sphere is placed in free space as shown. Calculate the electric field intensity at the center of the sphere (point P). Give magnitude and direction of the field.



Solution: Assume an element of volume at a general point (R',θ',ϕ') inside the half sphere as shown in **Figure A**. This element of volume is $dv'=R'^2 \sin\theta' dR' d\theta' d\phi'$. The element of charge due to this element of volume is $dq=\rho_0 R'^3 \sin\theta' dR' d\theta' d\phi'$. The electric field intensity, *dE*, due to this element of charge is:

$$dE = \frac{\rho_0 R'^3 \sin\theta' dR' d\theta' d\phi'}{4\pi\varepsilon_0 R'^2} = \frac{\rho_0 R' \sin\theta' dR' d\theta' d\phi'}{4\pi\varepsilon_0}$$

Because of symmetry above the horizontal, dashed line, only a horizontal field may exists, pointing to the left. This horizontal component, indicated as dE_h in the figure is:

$$dE_h = dE\cos\Psi = dE\sin\theta = \frac{\rho_0 R' \sin^2\theta' dR' d\theta' d\phi'}{4\pi\varepsilon_0}$$

Now we integrate over the half sphere:

$$E_h = \int_{R'=0}^{a}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{\pi} \frac{\rho_0 R' sin^2\theta' dR' d\theta' d\phi'}{4\pi\varepsilon_0} = \frac{\rho_0}{4\pi\varepsilon_0}\pi\int_{R'=0}^{a}\int_{\theta'=0}^{\pi} R' sin^2\theta' dR' d\theta'$$

$$= \frac{\rho_0}{4\varepsilon_0}\frac{a^2}{2}\int_{\theta'=0}^{\pi} R' sin^2\theta' d\theta' = \frac{\rho_0}{4\varepsilon_0}\frac{a^2}{2}\frac{\pi}{2} = \frac{\rho_0 \pi a^2}{16\varepsilon_0} \qquad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

2. A disk of radius a=1m is placed on the *x-y* plane and is charged with a uniform surface charge density $+\rho_s$ [C/m^2]. The disk has a total mass of w=10 grams and is fixed in place so it cannot move. A second, identical disk is placed above the first disk at a distance d=1mm, in free space. The second disk is also charged with a surface charge density $+\rho_s$ [C/m^2]. Calculate the required surface charge density so that the upper disk is suspended above the lower disk. That is, it will neither move upwards or fall down onto the lower disk. Specify the assumptions needed to solve this problem.

Solution: The disks will repel each other since they are both charged with positive charge density. For suspension to occur, the electric force must balance the weight of the upper plate.

From Gauss' law, the electric field between the plates is zero but the electric field intensity above the plates is:

$$\mathbf{E} = \hat{\mathbf{z}}\frac{\rho_s}{\varepsilon_0}$$

The electric force is QE and points upwards while the weight pulls the plate down and equals wg. Thus:

$$QE = wg$$

where Q is the total charge on the upper plate and E is the electric field on the plate (because the field is uniform on the plate). The total charge is the area of the plate, multiplied by the charge density ($Q=\pi a2\rho_s$):

$$QE = \left(\pi a^2\rho_s\right)\left(\frac{\rho_s}{\varepsilon_0}\right) = \frac{\pi a^2\rho_s^2}{\varepsilon_0} = wg$$

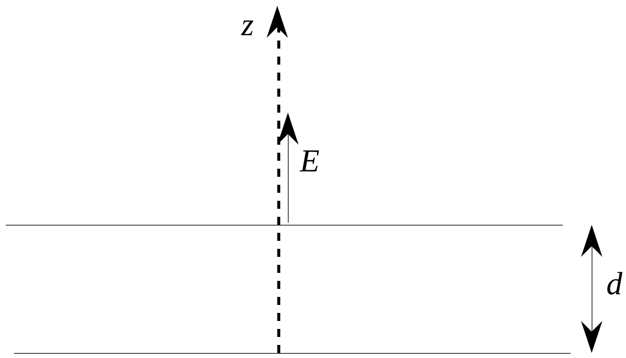
Thus:

$$\rho_s = \sqrt{\frac{\varepsilon_0 wg}{\pi a^2}} \qquad \left[\frac{\mathrm{C}}{\mathrm{m}^2}\right]$$

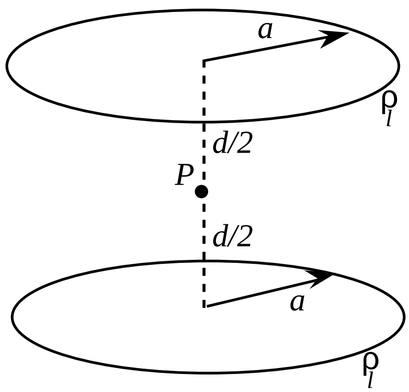
Or, numerically:

$$\rho_s = \sqrt{\frac{8.854\times10^{-12}\times10\times10^{-3}\times9.81}{\pi\times1^2}} = 5.258\times10^{-7} \qquad \left[\frac{\mathrm{C}}{\mathrm{m}^2}\right]$$

Assumption: Field is uniform on the plate (as if the plate were infinite in size). This is justified because the distance between the plates is small compared to the size of the plates.



3. Two conducting rings of radius a are both charged with an identical line charge density ρ_l [C/m] and are placed parallel to each other at a distance d apart as shown. What is the electric field intensity midway between the two rings (point P).



Answer: From symmetry considerations, the electric field intensity must be zero (both rings have the same sign charge and same total charge.

4. A very long coaxial cable has an inner conductor of radius a and outer conductor of radius b with a dielectric of permittivity ε between them. The electric field intensity has been measured exactly midway between the two conductors and was found to be equal to $\mathbf{E}=\hat{\mathbf{r}}E_0$.

a. Calculate the potential difference between the inner and outer conductor.
b. Calculate the capacitance per unit length of the coaxial cable.

Solution:
a. By use of Gauss' law, the electric field intensity anywhere between the conductors is:

$$\mathbf{E} = \hat{\mathbf{r}}\frac{\rho_l}{2\pi\varepsilon r}$$

where ρ_l is the charge density per unit length on the inner conductor. Since we need to calculate ρ_l, we set for the midway point between a and b:

$$E_0 = \frac{\rho_l}{2\pi\varepsilon(a+b)/2} \quad \rightarrow \quad \rho_l = E_0 2\pi\varepsilon(a+b)/2 = E_0\pi\varepsilon(a+b)$$

Now, we can write the general electric field intensity as:

$$\mathbf{E} = \hat{\mathbf{r}}\frac{\rho_l}{2\pi\varepsilon r} = \hat{\mathbf{r}}\frac{E_0\pi\varepsilon(a+b)}{2\pi\varepsilon r} = \hat{\mathbf{r}}\frac{E_0(a+b)}{2r}$$

To calculate the potential difference between inner and outer conductor, we write:

$$V_{ab} = -\int_b^a \mathbf{E}.d\mathbf{l} = -\int_b^a \hat{\mathbf{r}}\frac{E_0(a+b)}{2r}d\mathbf{r} = -\int_b^a \frac{E_0(a+b)}{2r}dr$$
$$= -\frac{E_0(a+b)}{2}\, ln\frac{a}{b} = \frac{E_0(a+b)}{2}\, ln\frac{b}{a} \qquad [\mathrm{V}]$$

This potential is positive since **E** points outwards (from inner to outer conductor).

b. The capacitance per unit length depends only on the dimensions and permittivity. This was calculated in class as:

$$C = \frac{2\pi\varepsilon}{ln\frac{b}{a}} \qquad [\mathrm{F}]$$