Electromagnetics I
Exam No. 1
September 30, 2005
Solution

Please read the exam carefully. Solve the following 4 problems. Each problem is 1/4 of the grade. To receive full credit, you must show all work. If you need to assume anything, state your assumptions clearly. Only reasonable assumptions that are necessary to solve the problem will be accepted. Any assumptions that modify the problem in any way will not be accepted. In all problems assume properties of free space (ε_0=8.85×10^{-12} F/m) unless otherwise indicated. You can write on both sides of the page. If you need additional space ask for additional paper and make sure you write your name on it. The exam is open book/notes.

Level of difficulty: 1 (easiest), 2, 3, 4 (most difficult)

e4.163. In chapter4.problems.extra 1. A sphere of radius a is made of a dielectric of permittivity ε and is placed in free space (free space has permittivity ε_0). The sphere contains a uniform charge density ρ_v, but we do not know the magnitude of this charge density. At a known distance $b>a$, the electric field intensity is known and given as $\mathbf{E} = -\hat{\mathbf{R}} E_0$. Calculate the volume charge density in the sphere.

Solution: Taking a Gaussian surface of radius b, which encloses the sphere symmetrically, we can write from Gauss' law:

$$E(4\pi b^2) = \frac{\rho_v\left(\frac{4}{3}\pi a^3\right)}{\varepsilon_0} \quad \rightarrow \quad E = \frac{\rho_v a^3}{3b^2\varepsilon_0}$$

Since we know that the electric field intensity at that point must equal E_0 in magnitude and must point in the negative R direction, we can write:

$$-\hat{\mathbf{R}}E_0 = \frac{\rho_v a^3}{3b^2\varepsilon_0} \quad \rightarrow \quad \rho_v = -\frac{3b^2\varepsilon_0 E_0}{a^3} \quad \left[\frac{\mathrm{C}}{\mathrm{m}^3}\right]$$

e4.164. In chapter4.problems.extra Very similar to e4.83. 2. An electric field in free space is given in the spherical system of coordinates by $\mathbf{E}(\mathbf{R}) = \hat{\mathbf{R}}2Pe^{-\alpha R}$ [V/m], where $\alpha > 0$ is a given numerical value and P is a given constant amplitude. Find:
(a) The electric potential of this field at the origin of the system of coordinates.
(b) The volume charge density at all points in space.

Solution: Given the electric field intensity, we calculate
a.

$$V(0) = -\int_{\infty}^{0} E(R)dR = -\int_{\infty}^{0} 2Pe^{-\alpha R}dR = \left.\frac{2Pe^{-\alpha R}}{\alpha}\right|_{\infty}^{0} = \frac{2P}{\alpha} \quad [\text{V}]$$

b. The charge density is calculated from the postulate (Gauss' law in differential form)

$$\nabla.\mathbf{E} = \frac{\rho_v}{\varepsilon} \quad \rightarrow \quad \rho_v = \varepsilon(\nabla.\mathbf{E}) = \frac{1}{R^2}\frac{d}{dR}\left(2PR^2e^{-\alpha R}\right) = \frac{2P\varepsilon}{R^2}\frac{d}{dR}\left(R^2e^{-\alpha R}\right)$$

$$= \frac{2P\varepsilon}{R^2}\left(-\alpha R^2e^{-\alpha R} + 2Re^{-\alpha R}\right) = 2P\varepsilon\left(-\alpha e^{-\alpha R} + \frac{2e^{-\alpha R}}{R}\right) = 2P\varepsilon e^{-\alpha R}\left(-\alpha + \frac{2}{R}\right) \quad \left[\frac{\text{C}}{\text{m}^3}\right]$$

e3.55 In chapter3.problems.extra. Almost identical to e3.28. 3. A flat washer as shown has a positive surface charge density $\rho_s = \rho_0$ uniformly distributed on its surface.

a. Calculate the electric field intensity at point *P* (on the axis, at a height *z* from the center of the ring).

b. Calculate the electric field intensity at the center of the ring (i.e. at $z = 0$).

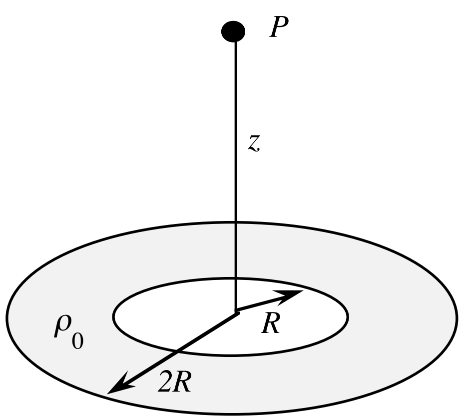

Figure A.

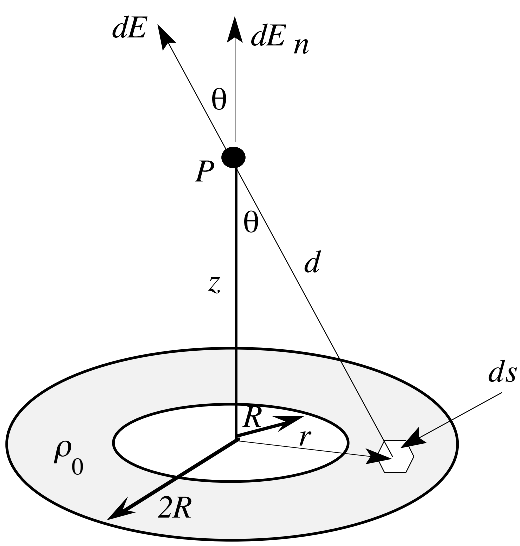

Figure B

Solution: Calculate the field due to an element of charge at P, calculate the vertical component (horizontal components cancel) and integrate over the ring (see Figure B).

a.

$$dE_n = \frac{\rho_0 r dr d\phi cos\theta}{4\pi\varepsilon_0 d^2} = \frac{\rho_0 r dr d\phi cos\theta}{4\pi\varepsilon_0 (r^2 + z^2)} = \frac{\rho_0 z r dr d\phi}{4\pi\varepsilon_0 (r^2 + z^2)^{3/2}}$$

$$E_n = \frac{\rho_0 z}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{r=R}^{2R}\frac{r dr d\phi}{(r^2 + z^2)^{3/2}} = \frac{\rho_0 z}{2\varepsilon_0}\int_{r=R}^{2R}\frac{r dr}{(r^2 + z^2)^{3/2}} = \frac{\rho_0 z}{2\varepsilon_0}\left[\frac{1}{\sqrt{R^2 + z^2}} - \frac{1}{\sqrt{4R^2 + z^2}}\right]$$

Or, in vector form:

$$\mathbf{E} = \hat{\mathbf{z}}\frac{\rho_0 z}{2\varepsilon_0}\left[\frac{1}{\sqrt{R^2 + z^2}} - \frac{1}{\sqrt{4R^2 + z^2}}\right] \qquad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

Note: there is no need to perform the integration.

b. The field at the center of the ring is zero. This can be done by inspection (due to symmetry) or by setting z = 0 in the result in a.

e4.165. In chapter4.problems.extra **4.** A solid cylinder of radius *a* may be assumed to be infinitely long. Two holes, each of radius *b* are drilled lengthwise as shown in cross-section in **Figure A**. The two holes are symmetric about the center of the cylinder and separated a distance *c* apart. A uniform volume charge density ρ_v is distributed throughout the remaining solid volume (gray area) but not in the holes. Assume the cylinder has permittivity ε while the permittivity in the holes and outside the cylinder is ε_0. Calculate the electric field intensity at the point *P*. Point *P* is at the surface of the cylinder in air.
Hint: Think of a superposition of charges that will produce the same field as the given configuration.

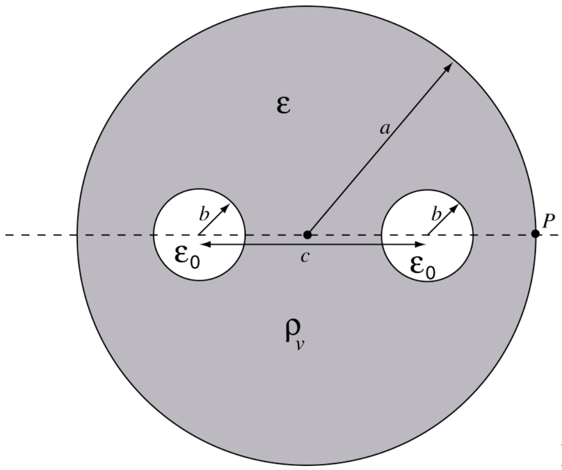

Figure A.

Solution:

Suppose first that we have a solid cylinder of radius a, without the holes with the same uniform charge density ρ_v. Now suppose that instead of the holes, we have two solid cylinders with a uniform charge density $-\rho_v$. The three cylinders combined produce the same exact charge distribution as in Figure A. The electric field intensity at P is the superposition of the three cylinders.

a. Due to the large cylinder we get at point P:

Using **Figure B**, the electric field intensity is (using a Gaussian surface of radius a):

$$E_1(2\pi aL) = \frac{\rho_v(\pi a^2 L)}{\varepsilon_0} \qquad \rightarrow \qquad E_1 = \frac{\rho_v(\pi a^2 L)}{(2\pi aL)\varepsilon_0} = \frac{\rho_v(\pi a^2)}{(2\pi a)\varepsilon_0}$$

Since this field is in the positive r direction, we have:

$$\mathbf{E}_1 = \hat{\mathbf{r}}\,\frac{\rho_v(\pi a^2)}{(2\pi a)\varepsilon_0} \qquad \left[\frac{\text{V}}{\text{m}}\right]$$

b. Due to the two smaller, symmetric cylinders, which have negative charge density we have (see **Figure C**):

Due to the right hand cylinder, again using a Gaussian surface centered at the small cylinder (c/2 to the right of the center of the large cylinder):

$$E_2(2\pi(a-c/2)L) = \frac{-\rho_v(\pi b^2 L)}{\varepsilon_0} \qquad \rightarrow \qquad E_2 = -\frac{\rho_v(\pi b^2)}{(2\pi(a-c/2)\varepsilon_0}$$

Due to the left hand cylinder, again using a Gaussian surface centered at the left small cylinder (c/2 to the left of the center of the large cylinder):

$$E_3(2\pi(a+c/2)L) = \frac{-\rho_v(\pi b^2 L)}{\varepsilon_0} \qquad \rightarrow \qquad E_3 = -\frac{\rho_v(\pi b^2)}{(2\pi(a+c/2)\varepsilon_0}$$

Both of these fields point in the negative r direction as shown. The are therefore:

$$\mathbf{E}_2 = -\hat{\mathbf{r}}\frac{\rho_v(\pi b^2)}{(2\pi(a - c/2)\varepsilon_0} \qquad \mathbf{E}_3 = -\hat{\mathbf{r}}\frac{\rho_v(\pi b^2)}{(2\pi(a + c/2)\varepsilon_0} \qquad \left[\frac{\text{V}}{\text{m}}\right]$$

The total field at point P is the sum of these three fields:

$$\mathbf{E}_P = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 = \hat{\mathbf{r}}\frac{\rho_v(\pi a^2)}{(2\pi a)\varepsilon_0} - \hat{\mathbf{r}}\frac{\rho_v(\pi b^2)}{(2\pi(a - c/2)\varepsilon_0} - \hat{\mathbf{r}}\frac{\rho_v(\pi b^2)}{(2\pi(a + c/2)\varepsilon_0}$$
$$= \hat{\mathbf{r}}\frac{\rho_v}{2\varepsilon_0}\left(a - \frac{b^2}{a - c/2} - \frac{b^2}{a + c/2}\right) \qquad \left[\frac{\text{V}}{\text{m}}\right]$$

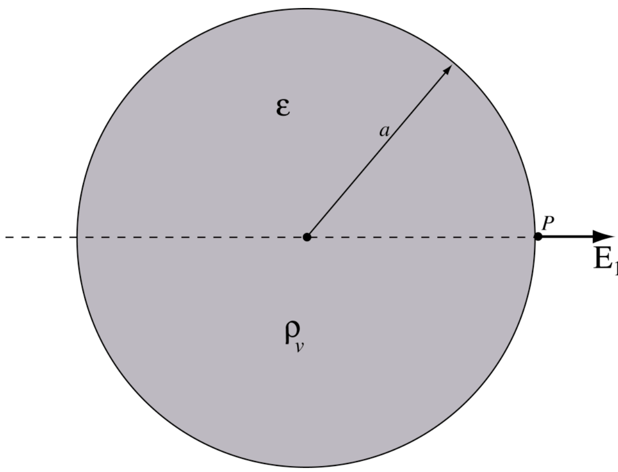

Figure B

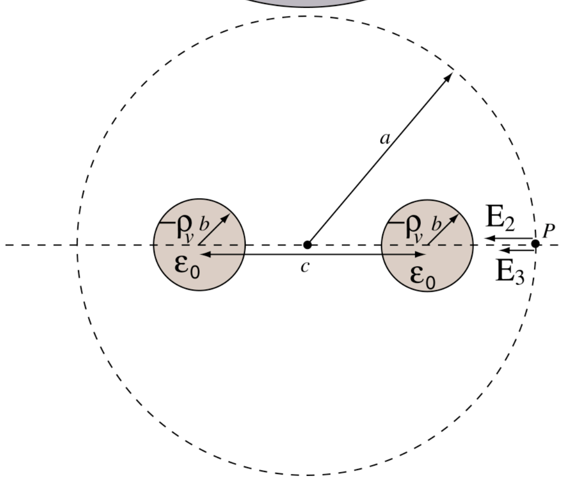

Figure C