Summary   Chapter 9

The current chapter applies the Biot-Savart and Ampere's laws and introduces most of the concepts in magnetostatics including inductance, energy and interface conditions. We start with the magnetic flux density and the dipole moment of a small loop of radius d carrying current I, at large distances ($R>>d$), (see **Figure 9.1**)

$$\mathbf{B} \approx \frac{\mu_0 m}{4\pi R^3}\left(\hat{\mathbf{R}} 2\cos\theta + \hat{\theta}\sin\theta\right) \quad [\mathrm{T}] \quad (9.14)$$

$\mathbf{m} = \hat{\mathbf{n}} I \pi d^2$ [A.m$_2$] is the **magnetic dipole moment** ($\hat{\mathbf{n}}$ is the normal to the loop).

The dipole moment leads to the definition of **magnetization** and magnetization current density. The magnetization **M** is due to a magnetization volume current density $\mathbf{J}_m$ and a magnetization surface current density $\mathbf{J}_{ms}$:

$$\mathbf{J}_m = \nabla \times \mathbf{M} \quad \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right], \qquad \mathbf{J}_{ms} = \mathbf{M} \times \hat{\mathbf{n}} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right] \quad (9.28)$$

The magnetization manifests itself in the permeability of the material

$$\mathbf{B} = \mu_0 \mathbf{H}_e + \mu_0 \mathbf{M} = \mu_0 \mu_r \mathbf{H}_e \quad [\mathrm{T}] \quad (9.30) \text{ through } (9.34)$$

where $\mu = \mu_0 \mu_r$ [H/m] is the magnetic permeability of the material and μ_r [dimensionless] its relative permeability. The higher the magnetization, the higher the permeability of the medium.

Magnetic materials – properties

1. **Diamagnetic materials** are materials with relative permeability slightly smaller than 1
2. **Paramagnetic materials** have relative permeability slightly higher than 1
3. **Ferromagnetic materials** are characterized by very high permeability ($\mu_r >> 1$)
4. Ferromagnetic materials exhibit **hysteresis** – a nonlinear effect due to magnetic domains whereby the relation between the magnetic flux density and magnetic field intensity under ac conditions follows a closed path (**Figure 9.16**)
5. Hysteresis is responsible for losses but also for the existence of permanent magnets – magnetized materials that retain their magnetization
6. **Soft magnetic materials** are those materials that can be easily demagnetized
7. **Hard magnetic materials** are "hard" to demagnetize and are used for production of permanent magnets.

Interface conditions for the magnetic field defines the behavior at interfaces. The magnetic interface conditions between two materials are (see **Eqs. (9.38)** through **(9.44)** and **Table 9.5**)

$$\hat{\mathbf{n}} \times \left(\mathbf{H}_{1t} - \mathbf{H}_{2t}\right) = \mathbf{J}_s, \quad \hat{\mathbf{n}} \times \left(\frac{\mathbf{B}_{1t}}{\mu_1} - \frac{\mathbf{B}_{2t}}{\mu_2}\right) = \mathbf{J}_s \quad \text{and} \quad B_{1n} = B_{2n}, \quad \mu_1 H_{1n} = \mu_2 H_{2n}$$

$\hat{\mathbf{n}}$ points into material (1) and a surface current density $\mathbf{J}_s$ [A/m] may exist at the interface between conductors and nonconductors (see **Figure 9.17**).

Inductance is the ratio of flux linkage and the current that produces it. It is independent of current and only depends on physical dimensions and permittivity. Given two circuits made of N_1 and N_2 loops and carrying currents I_1 and I_2 respectively, we define **self inductances** L_{11}, L_{22} and **mutual inductances** L_{12}, L_{21} as (see **Eqs. (9.56)** through **(9.59)**):

$$L_{11}=\frac{N_1\Phi_{11}}{I_1}=\frac{\Lambda_{11}}{I_1},\quad L_{22}=\frac{N_2\Phi_{22}}{I_2}=\frac{\Lambda_{22}}{I_1},\quad L_{12}=\frac{N_2\Phi_{12}}{I_1}=\frac{\Lambda_{12}}{I_1},\quad L_{21}=\frac{N_1\Phi_{21}}{I_2}=\frac{\Lambda_{21}}{I_1}\quad [\mathrm{H}]$$

Φ_{11} is the flux produced by circuit (1) linking all turns of circuit (1), Φ_{12} is the flux produced by circuit (1) and linking with all turns of circuit (2) and so on. $\Lambda=N\Phi$ is called **flux linkage**.

Inductance entails assumption of a current in a circuit, calculation of the flux density, calculation of flux and flux linkage followed by division by the current that generated the flux. However, the inductance is independent of the assumed current. In infinite structures a more useful relation is self- and mutual-inductance per unit length of the device.

External inductance – inductance due to flux outside conductors

Internal inductance – inductance due to flux within the conductor's volume.

Energy stored in the magnetic field is closely related to inductance even where inductors cannot be clearly identified. The basic definition starts with the energy stored in an inductor L due to passage of current I

$$W_m=\frac{LI^2}{2}\quad [\mathrm{J}]\qquad (9.62)$$

In a system of n loops or coils, each with N_i turns and current I_i, the magnetic energy is

$$W_m=\frac{1}{2}\sum_{i=1}^{n}N_i\Phi_i I_i\quad [\mathrm{J}]\qquad (9.72)$$

where Φ_i is the total flux in loop i due to all loops in the system. Alternatively it may be written in terms of inductance as

$$W_m=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}I_iI_j\quad [\mathrm{J}]\qquad (9.77)$$

where L_{ij} is the self inductance ($i=j$) or mutual inductance ($i\neq j$) between circuits i,j, and the current I_i,I_j are the currents in the circuits. If the currents I_i,I_j produce fluxes in the same direction, L_{ij} is considered positive, if not, negative. A more general approach is in terms of fields. This is particularly useful when inductances cannot be identified and calculated such as in space. The magnetic energy can be calculated from various field quantities as follows:

$$W_m=\frac{1}{2}\int_{v'}\mathbf{A}\cdot\mathbf{J}\,dv'=\frac{1}{2}\int_{v}\mathbf{B}\cdot\mathbf{H}\,dv=\frac{1}{2}\int_{v}\mu H^2\,dv=\frac{1}{2}\int_{v}\frac{B^2}{\mu}dv\quad [\mathrm{J}]\qquad (9.84), (9.90)$$

v' here is the volume in which the current density **J** exists whereas v is the volume in which the magnetic field is nonzero. The first of these is useful where **J** is easily identified. In space, where currents may not exist, calculation in terms of the magnetic flux density and field intensity is used.

The integrand in the energy expressions is the energy density

$$w_m=\frac{\mathbf{A}\cdot\mathbf{J}}{2}=\frac{\mathbf{B}\cdot\mathbf{H}}{2}=\frac{BH}{2}=\frac{\mu H^2}{2}=\frac{B^2}{2\mu}\quad \left[\frac{\mathrm{J}}{\mathrm{m}^3}\right]\qquad (9.91)$$

Magnetic circuits are based on the equivalence between electric currents in closed circuits and flux in closed magnetic circuits. The requirements are for permeability to be high and any gaps in the circuits to be as small as possible to avoid flux leakage around the gaps. Under these conditions the flux in any closed loop within the magnetic circuit is

$$\Phi = \frac{\sum_{i=1}^{n} N_i I_i}{\sum_{j=1}^{k} \Re_j} \quad (9.102), \text{ with } \Re_j = \frac{l_j}{\mu_j S_j} \quad \left[\frac{1}{\text{H}}\right] \quad (9.99)$$

where N_i is the number of turns in coil i, I_i its current and $\Re_j$ is the magnetic reluctance of segment j of the magnetic path with l_j the length of the segment, μ_j its permeability and S_j its cross-sectional area (see **Figure 9.37**). Magnetic circuits allow simple calculation of fluxes and magnetic fields in devices which satisfy the basic requirements of a magnetic circuit.

Forces in the magnetic field are defined based on the Lorentz force equation ($\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$) in **Eq. (9.106)** which governs forces on moving charges. Since moving charges constitute currents, **Eq. (9.106)** can be developed into more useful relations. For currents in a volume we have:

$$\mathbf{F}_m = \int_{v'} \mathbf{J} \times \mathbf{B} dv' \quad [\text{N}] \quad (9.110)$$

where $\mathbf{J} \times \mathbf{B}$ is a volume force density [N/m^3]. In thin wires carrying current I

$$\mathbf{F}_m = \int_{L'} I\, d\mathbf{l}' \times \mathbf{B} \quad [\text{N}] \quad (9.113)$$

In all cases, the flux density **B** is due to sources other than I (I cannot cause a force on itself).

Eq. (9.113) is the basis of Neumann's formula which defines a force between any two current carrying wire segments that are part of closed circuits (see **Figure 9.45)**:

$$\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \int_{p_3}^{p_4} \int_{p_1}^{p_2} \frac{d\mathbf{l}_2' \times \left(d\mathbf{l}_1' \times \hat{\mathbf{R}}_{12}\right)}{R_{12}^2} \quad [\text{N}] \quad (9.117)$$

Here one segment extends between p_1 and p_2, the second between p_3 and p_4 and the vector $\mathbf{R}_{12}$ connects $d\mathbf{l}_1'$ and $d\mathbf{l}_2'$. Both **Eqs. (9.113)** and **(9.117)** can be extended for closed loops (see **Eqs. (9.114)** and **(9.118)**).

Principle of virtual work. Given the magnetic energy in a system, a force, such as between two faces of a gap, may be calculated by allowing a virtual displacement of one of the faces. The change in energy is related to force (**Section 9.7.1**):

$$F = -\nabla W \quad [\text{N}] \quad (9.121)$$

Any time there is a force on a system, there may also be a torque. **Torque** is simply the force multiplied by arm-length. In the case of a loop with magnetic dipole moment **m**, the torque may be written as

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \quad [\text{N} \cdot \text{m}] \quad (9.127)$$