Summary Chapter 6.

Numerical methods of solution are used when the analytical methods in Chapters 3 through 5 fail, usually because the geometry is too complicated. The three methods described in this chapter are representative of the concepts involved.

The **finite difference method** replaces partial or ordinary derivatives with simple approximations. Given points x_i on a line, at distance $\Delta x=h$ from each other, with unknown values $f(x_i)=f_i$, the approximation to first and second order derivatives may be written as

$$\frac{df(x_i)}{dx} \approx \frac{f(x_i+\Delta x)-f(x_i-\Delta x)}{2\Delta x} = \frac{f_{i+1}-f_{i-1}}{2h} \quad (6.1), \qquad \frac{d^2 f_i}{dx^2} \approx \frac{f_{i+1}-2f_i+f_{i-1}}{h^2} \quad (6.8)$$

Similar expression are written for derivatives with respect to y and z if needed. To use these approximations, the space in which a solution is sought is divided into a grid with points $i,j,$ generated by parallel lines, separated distances $h=\Delta x=\Delta y$, in the two directions in space forming a two dimensional grid (**Figure 6.5**). An unknown potential $V_{i,j}$, is assumed at each point of the grid and using the approximation in Eq. (6.8) we get:

$$V_{i,j} = \frac{V_{i-1,j}+V_{i+1,j}+V_{i,j-1}+V_{i,j+1}}{4} + h^2 \frac{\rho_{i,j}}{4\varepsilon_{i,j}} \quad (6.17)$$

This equation is repeated at each internal point of the grid. Boundary values are incorporated in the approximation when any of the values of V corresponds to a boundary point. Solution of the system of equations provides the unknown values over the grid. Other quantities such as electric fields, forces, and energy can then be calculated from the potential. The extension to three dimensions is straightforward (**Exercise 6.3**). The charge density at points of the grid can vary from point to point as can the permittivity.

The **method of moments** solves for the equivalent sources that produce a given potential. Given a charge distribution $\rho_{\Omega'}(x',y',z')$, the potential at a distance R from a differential point charge is

$$V(x,y,z) = \int_{\Omega'} \frac{1}{4\pi\varepsilon R}\rho_{\Omega'}(x',y',z')d\Omega' = \int_{\Omega'} K(x,y,z,x',y',z')\rho_{\Omega'}(x',y',z')d\Omega' \quad (6.23) \text{ and } (6.24)$$

$K(x,y,z,x',y',z')$ is a geometric function that depends on dimensions and permittivity and Ω' is the space in which the charge density is distributed (surface, volume). To apply the method, the space (say a plate) is divided into any number of subspaces (subsurfaces), an unknown charge density is assumed on each subsurface whereas the potential on the surface must be known. The geometric function K is calculated for each pair of subsurfaces assuming the charge is concentrated at its center. Assuming the surface has been divided into N rectangular subsurfaces, each with an area $s_i=a_i \times b_i$ the known potential on subsurface j is

$$V_j(x_j,y_j,z_j) = \sum_{i=1}^{N} \rho_j K_{i,j}, \qquad j=1,2,...,N \quad (6.28)$$

where

$$K_{i,j} = \frac{s_i}{4\pi\varepsilon\sqrt{\left(x_j-x_i'\right)^2+\left(y_j-y_i'\right)^2+\left(z_j-z_i'\right)^2}}, \qquad i,j=1,2,...,N, \ i\neq j \quad (6.29)$$

$$K_{jj} = \frac{1}{4\pi\varepsilon}\left(2a_j \ln\frac{b_j+\sqrt{a_j^2+b_j^2}}{a_j} + 2b_j \ln\frac{a_j+\sqrt{a_j^2+b_j^2}}{b_j}\right) \quad (6.31$$

Eq. (6.28) is written for the potential on each subdomain to obtain N equations in N unknowns. Solution of this system provides the charge density on each subdomain from which we can then calculate potentials and fields anywhere in space using the methods of Chapters 3 and 4.
Notes:

1. The larger the number of subdomains the more accurate the solution
2. Subdomains can be of different sizes or all equal in size
3. The potential must be known on each subdomain
4. When solving for the charge densities on plates of capacitors, the total charge on one plate must equal in magnitude to the total charge on the second. The potential of each plate must be adjusted to satisfy this condition (see **Example 6.7**)

The **finite element method** assumes the space (line, surface, volume) is divided into finite size sections or elements and a potential distribution (constant, linear, quadratic, etc.) is assumed within each element based on the nodes (vertices) of the element. The potentials at these vertices are the unknowns we seek. The approximation for each element is then used to generate a system of equations for the unknown potentials. Given a triangular element with vertices (x_i,y_i), the potential within the element is

$$V(x,y) = N_i V_i + N_j V_j + N_k V_k \qquad (6.46)$$

where N_i, N_j, N_k are called shape functions and V_i, V_j, V_k the unknown potentials at the nodes of the element. For a triangular element of area Δ and node coordinates $(x_i y_i,)$, $(x_j,y_j,)$ and $(x_k y_k)$:

$$N_i = \frac{1}{2\Delta}\left[\left(x_j y_k - x_k y_j\right) + \left(y_j - y_k\right)x + \left(x_k - x_j\right)y\right] \qquad (6.47)$$

$$N_j = \frac{1}{2\Delta}\left[\left(x_k y_i - x_i y_k\right) + \left(y_k - y_i\right)x + \left(x_i - x_k\right)y\right] \qquad (6.48)$$

$$N_k = \frac{1}{2\Delta}\left[\left(x_i y_j - x_j y_i\right) + \left(y_i - y_j\right)x + \left(x_j - x_i\right)y\right] \qquad (6.49)$$

To solve an electrostatic problem, we write the energy in a volume as

$$F(E) = \int_v \left(\frac{1}{2}\varepsilon E 2 - \rho V\right) dv \qquad (6.66)$$

For the two dimensional problems discussed in this chapter, assuming unit thickness for the geometry (dv=1ds) and using $\mathbf{E} = -\nabla V$:

$$F(V) = \int_s \left(\frac{\varepsilon}{2}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right\} - \rho V\right) ds \qquad (6.69)$$

This energy is minimized with respect to each unknown value in each element to produce a solution

$$\frac{\partial F(V)}{\partial V_n} = \sum_{m=1}^{M} \frac{\partial F_m}{\partial V_n} = 0 \qquad (6.71)$$

M is the number of elements in the assembly (called a mesh) and N the number of nodes. **Eq. (6.71)** produces N equations in N unknowns which, when boundary conditions are applied and the system solved, produces the potentials at the nodes of the finite element mesh.
The solution consists of the following

1. Definition of a finite element (triangle in two dimensions, tetrahedron in three dimensions or any other defined shape that divides the space)

2. Approximation of the potential over the element
3. The equation to solve is an energy related function (**Eq. (6.69)** for example)
4. Minimization of the energy function over the space
5. Application of boundary conditions and solution for potentials
6. Electric fields can then be calculated from potentials if necessary (see **Example 6.10**).