Summary Chapter 5.

Chapter 5 continues discussion of methods of solution for the electrostatic field. The methods are called collectively boundary value problems because they essentially solve Poisson's (or Laplace's) equation under given boundary conditions.

Poisson's equation for the electric potential is the basis of the methods that follow:

$$\nabla^2 V = -\frac{\rho v}{\varepsilon}, \quad \text{in Cartesian coordinates:} \quad \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = -\frac{\rho v}{\varepsilon} \qquad (5.4) \text{ and } (5.5)$$

If the right hand side is zero, we get **Laplace's equation**. The equation can be written in cylindrical and spherical coordinates as well (**Eqs. (5.6)** and **(5.7)**). Solution provides the potential and, if necessary, the electric field intensity. **Direct integration** of either Poisson's or Laplace's Equations is possible if the potential only depends on one variable (one-dimensional problem). The process is as follows:

1. Integrate twice to obtain the potential as a function of the variable
2. Substitute known values of the potential (boundary conditions) to obtain the constants of integration
3. Substitute the constants into the function in (1) to obtain the particular solution for V.

Method of images. The method solves for fields and potentials due to charges (point charges or charge distributions) in the presence of conductor(s) by removing the conductor(s) and replacing the conducting surface(s) with image charges that maintain the potential on the surface(s) unchanged. Then the solution outside the conductor(s) is due to the original charges and the image charges.

Flat conducting surfaces

1. Image charges are equal in magnitude to the original charges, opposite in signs and placed at the same distance below the surface
2. The number of image charges depends on the number of conducting surfaces
3. For two parallel conducting plates with point charges between them, the number of image charges is infinite
4. For conducting surfaces at an angle α; If $n=180^\circ/\alpha$ is an integer, the number of image charges is $2n-1$. If n is not an integer, the method cannot be used (see **Example 5.6**).

Lines of charge or volume charge distributions in the presence of conductors behave the same way as point charges since they are assemblies of point charges.

Cylindrical surfaces: lines of charge parallel to conducting cylindrical surfaces reflect as follows:

1. The image line of charge is parallel and equal in magnitude to the original line of charge. The sign of the image line of charge is opposite (**Figure 5.23**)
2. A line of charge at a distance d outside a cylindrical conductor of radius $a<d$ produces an image charge at a distance b from the center of the conductor such that

$$b = \frac{a^2}{d} \quad [\text{m}] \quad (5.35)$$

Point charges and conducting spherical surfaces

A point charge q at a distance d from the center of a conducting sphere of radius $a<d$ creates an image with magnitude q' inside the sphere at a distance b from its center (**Figure 5.26**):

$$b = \frac{a^2}{d} \quad [\text{m}], \qquad q' = \frac{qa}{d} \quad [\text{C}] \qquad (5.42)$$

Notes:

1. In all cases, the images do not actually exist – they are artificially postulated for calculation purposes. They replace the effect of conductors
2. The fields found are only valid outside conductors
3. The potential of the conducting surface is assumed to be zero

4. The effect of any other potential, charge or charge distribution must be added to the solution found by the method of images.

Separation of variables

Laplaces equation is a second order differential equation and can be solved using the method of separation of variables in any system of coordinates. The general solution is:
In Cartesian coordinates (3-dimensions):

$$V(x,y,z)=\left[A_1\sin(k_x x)+A_2\cos(k_x x)\right]\left[A_3\sin(k_y y)+A_4\cos(k_y y)\right]\left[A_5\sin(|k_z|z)+A_6\cos(|k_z|z)\right] \quad [\text{V}] \quad (5.59)$$

or

$$V(x,y,z)=\left[B_1e^{jk_x x}+B_2e^{-jk_x x}\right]\left[B_3e^{jk_y y}+B_4e^{-jk_y y}\right]\left[B_5e^{j|k_z|x}+B_6e^{-j|k_z|x}\right] \quad [\text{V}] \quad (5.60)$$

where $k_x^2+k_y^2+k_z^2=0$ (5.52)

In Cartesian coordinates (2-dimensions in *x-y*), with $k_z^2=-k_x^2$:

$$V(x,z)=\left[A_1\sin(k_x x)+A_2\cos(k_x x)\right]\left[A_5\sinh(|k_z|z)+A_6\cosh(|k_z|z)\right] \quad [\text{V}] \quad (5.61)$$

$$V(x,z)=\left[B_1e^{jk_x x}+B_2e^{-jk_x x}\right]\left[B_5e^{j|k_z|x}+B_6e^{-j|k_z|x}\right] \quad [\text{V}] \quad (5.62)$$

In 2-dimensional cylindrical coordinates (variation in the r-ϕ plane, no variation of the field in the z-direction):

$$V(r,\phi)=\left[A_1\sin(k\phi)+A_2\cos(k\phi)\right]\left[B_3r^k+B_4r^{-k}\right] \quad [\text{V}] \quad (5.75)$$

or:

$$V(r,\phi)=\left[A_1e^{jk\phi}+A_2e^{-jk\phi}\right]\left[B_3r^k+B_4r^{-k}\right] \quad [\text{V}] \quad (5.76), \qquad k^2=k_\phi^2=-k_r^2$$