

Summary Chapter 4.

Chapter 4 continues the discussion on the electrostatic fields. We start with the postulates – the definition of the divergence and curl followed by their integral representation:

$$\overbrace{\nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon_0}, \quad \nabla \times \mathbf{E} = 0}^{\text{differential form}} \quad \text{or} \quad \overbrace{\oint_s \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}, \quad \oint_l \mathbf{E} \cdot d\mathbf{l} = 0}^{\text{integral form}} \quad (4.10)$$

The first of these (divergence equation) is Gauss's law. It is particularly useful in its integral form for evaluation of the electric field intensity or electric flux density as follows:

$$\oint_s \mathbf{E} \cdot d\mathbf{s} = \frac{Q_{enc.}}{\epsilon_0} \quad (4.13) \quad \text{or} \quad \oint_s \mathbf{D} \cdot d\mathbf{s} = Q_{enc.} \quad (4.12)$$

where $Q_{enc.}$ is the total charge enclosed by the surface s :

Gauss's law is universally applicable for calculation of fields from charge distributions or of the equivalent charge from fields. For practical use we require the electric field intensity \mathbf{E} (or electric flux density \mathbf{D}) to be constant in magnitude and either perpendicular or tangential to a surface called **Gaussian surface** which encloses a total charge $Q_{enc.}$ symmetrically. This allows us to take the quantities \mathbf{D} or \mathbf{E} outside the integral sign after performing the scalar product and evaluate them. Note:

1. Only highly symmetric charge configuration can be treated analytically
2. The magnitude but not the direction of the field is found
3. When Gauss' law is used for calculation of charges from known electric fields, only the total equivalent charge enclosed by the Gaussian surface is calculable, not its actual distribution.

The **electric potential**, is defined as the difference in potential energy per unit charge. This establishes the potential difference between two arbitrary points a, b due to an electric field intensity \mathbf{E} as:

$$V_{ba} = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = V_b - V_a \quad \left[\frac{\text{J}}{\text{C}} \right] \text{ or } \left[\frac{\text{N} \cdot \text{m}}{\text{C}} \right] \text{ or } [\text{V}] \quad (4.20)$$

The absolute potential at a point is the potential difference between that point and a zero reference point, usually, but not always, at infinity. A system of n point charges produces a potential at a position vector \mathbf{R} as

$$V(\mathbf{R}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{Q_i}{|\mathbf{R} - \mathbf{R}_i|} \quad [\text{V}] \quad (4.29)$$

By defining differential point charges within line, surface and volume charge distributions, **Eq. (4.29)** extends to potentials of any charge distribution (see **Eqs. (4.31)** through **(4.33)**):

$$V(\mathbf{R}) = \frac{1}{4\pi\epsilon_0} \int_{\Omega} \frac{\rho_{\Omega'}}{|\mathbf{R} - \mathbf{R}'|} d\Omega' \quad [\text{V}], \quad \text{where } \Omega' \text{ stands for } l', s' \text{ or } v'.$$

The electric field intensity may be evaluated from the potential as (see Eqs. Xxx through yyy)

$$\mathbf{E} = -\nabla V \quad [\text{V/m}] \quad (4.35)$$

This method requires calculation of V in general coordinates but is particularly useful because calculation of potential, being a scalar function, is often easier than direct calculation of \mathbf{E} .

Materials: Perfect conductors and dielectrics

In **perfect conductors**:

1. The electric field intensity is zero inside the medium
2. The electric field intensity is everywhere normal to its surface
3. A surface charge density must exist equal in magnitude to $\rho_s = \epsilon E_n$. The sign of the charge is defined by the direction of the normal electric field intensity E_n .

Dielectrics are insulating (non-conducting) media. We restrict ourselves here to perfect dielectrics.

Polarization in dielectrics is due to the effect of the external electric field on the atoms of the medium and manifests itself through the existence of surface and volume polarization charge densities

$$\rho_{ps} = \mathbf{P} \cdot \hat{\mathbf{n}}' \quad \left[\frac{\text{C}}{\text{m}^2} \right] \quad \text{and} \quad \rho_{pv} = \nabla' \cdot \mathbf{P} \quad \left[\frac{\text{C}}{\text{m}^3} \right] \quad (4.52)$$

A polarization vector \mathbf{P} , may be postulated either as a sum of the polarization fields of individual atoms (**Eq. (4.38)**) or from the macroscopic effect on the external electric flux density:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon_r \mathbf{E} \quad \left[\frac{\text{C}}{\text{m}^2} \right] \quad (4.59) \quad \text{and} \quad (4.64)$$

The **relative permittivity** ϵ_r [non-dimensional] incorporates the effects of polarization and is a property of the dielectric. It is the ratio between the permeability of the material and that of free space.

Dielectric strength is the maximum electric field intensity, beyond which the dielectric breaks down – essentially conducts current. Each dielectric has a given, measurable dielectric strength. The dielectric strength in air is approximately 3000 [V/mm].

Interface conditions describe the relations between the electric field on the two sides of the interface between different materials (see **Figures 4.30** and **4.31**). See **Table 4.3** for a summary.

$$E_{1n} = E_{2n}, \quad \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2} \quad \text{and} \quad D_{1n} - D_{2n} = \rho_s, \quad \epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s \quad (4.69), (4.70) \quad \text{and} \quad (4.76)$$

The surface charge density ρ_s is nonzero on conductors.

Capacitance is the property of a body to store charge when connected to a potential. As the ratio between charge and potential it applies primarily to conducting bodies:

$$C = \frac{Q}{V} \quad [\text{F}] \quad (4.83)$$

1. The capacitance of an isolated body is the charge on that body divided by its potential and may be viewed as the capacitance between the body and infinity
2. The capacitance between two bodies is the charge on one body divided by the potential difference between them
3. Capacitance is calculated by either assuming a known charge or charge density on the bodies and calculating the resulting potential or assuming a known potential difference and calculating the charge
4. In either case capacitance is independent of charge or potential – it only depends on physical dimensions and on permittivity.

For parallel plate capacitors with plates of area A , separation d and permittivity ϵ between the plates:

$$C = \frac{\epsilon A}{d} \quad [\text{F}] \quad (4.90)$$

The capacitance of N capacitors connected in series or in parallel is calculated as follows:

$$\text{Series: } \frac{1}{C_{total}} = \sum_{i=1}^N \frac{1}{C_i} \quad [\text{F}] \quad (4.98), \quad \text{Parallel: } C_{total} = \sum_{i=1}^N C_i \quad [\text{F}] \quad (4.94)$$

Energy in the electrostatic field is defined from potential since the latter is potential energy per unit charge. Given a system of N point charges, the energy in the system is

$$W = \frac{1}{2} \sum_{i=1}^N Q_i V_i \quad [\text{J}] \quad (4.112)$$

By defining differential line, surface and point charges, the definition of energy can be extended to charge distributions as (see **Eqs. (4.116)** through **(4.118)**):

$$W = \frac{1}{2} \int_{\Omega'} \rho_{\Omega'} V d\Omega' \quad [\text{J}] \quad (\Omega' \text{ stands for } l', s' \text{ or } v')$$

The potential may be written in terms of the electric field intensity as well:

$$W = \frac{1}{2} \int_v \mathbf{D} \cdot \mathbf{E} dv = \frac{1}{2} \int_v \epsilon E^2 dv = \frac{1}{2} \int_v \frac{D^2}{\epsilon} dv \quad [\text{J}] \quad (4.126)$$

Integration is over the space in which the electric field intensity is nonzero.

The integrands in **Eq. (4.126)** are the energy densities. These may be written as follows:

$$w = \frac{\mathbf{D} \cdot \mathbf{E}}{2} = \frac{\epsilon E^2}{2} = \frac{D^2}{2\epsilon} = \frac{\rho_v V}{2} \quad \left[\frac{\text{J}}{\text{m}^3} \right] \quad (4.127), (4.128) \text{ and } (4.129)$$

Since energy may be viewed as integrated force over distance it is also possible to calculate the force in terms of energy as:

$$\mathbf{F} = -\nabla W_e \quad (4.136)$$