Antenna Theory
Exam No. 1
October 9, 2000

Solve the following 4 problems. Each problem is 20% of the grade. To receive full credit, you must show all work. If you need to assume anything, state your assumptions clearly. Reasonable assumptions that are necessary to solve the problem will be accepted. In all problems assume properties of free space (ε_0=8.854×10^{-12} F/m, μ_0=4π×10^{-7} H/m, η_0=377 Ω)

1. A one wavelength, thin wire dipole antenna is given. Assuming a peak current of 1A at 600 MHz calculate:
a. The radiated power of the antenna.
b. The radiation resistance of the antenna.
c. The directivity and maximum directivity of the antenna

Solution: Find the far-field electric and magnetic field intensities. Calculate the time averaged power density and integrate it over a sphere of radius R to obtain the radiated power. From radiated power and current, calculate the radiation resistance. From radiated power and power density calculate the radiation intensity and then the directivity of the antenna.

From **Eqs. (18.85)** and **(18.86)**:

$$\mathbf{E} = \hat{\theta}\frac{j\eta I_0}{2\pi R}e^{-j\beta R}\frac{cos\left((\beta L/2)cos\theta\right) - cos(\beta L/2)}{sin\,\theta}$$

$$\mathbf{H} = \hat{\varphi}\frac{jI_0}{2\pi R}e^{-j\beta R}\frac{cos\left((\beta L/2)cos\theta\right) - cos(\beta L/2)}{sin\,\theta}$$

The time averaged poer density is:

$$\mathbf{P}_{av} = \frac{\mathbf{E}\times\mathbf{H}^*}{2} = \left(\hat{\theta}\times\hat{\varphi}\right)\left(\frac{jI_0}{2\pi R}e^{-j\beta R}\frac{cos\left((\beta L/2)cos\theta\right) - cos(\beta L/2)}{sin\,\theta}\right)\left(\frac{-j\eta I_0}{2\pi R}e^{j\beta R}\frac{cos\left((\beta L/2)cos\theta\right) - cos(\beta L/2)}{sin\,\theta}\right)$$
$$= \hat{\mathbf{R}}\frac{I_0^2\eta}{8\pi^2R^2}\left(\frac{cos\left((\beta L/2)cos\theta\right) - cos(\beta L/2)}{sin\,\theta}\right)^2$$

With L=λ and β=2π/λ we get βL/2=π. Thus:

$$\mathbf{P}_{av} = \hat{\mathbf{R}}\frac{I_0^2\eta}{8\pi^2R^2}\left(\frac{cos(\pi cos\theta) + 1}{sin\theta}\right)^2 \qquad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

Integrating over a spherical surface at radius R:

$$P_{rad} = \oint_s \mathbf{P}_{av}.d\mathbf{s} = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{I_0^2\eta}{8\pi^2R^2}\left(\frac{cos(\pi cos\theta) + 1}{sin\theta}\right)^2 R^2 sin\theta d\theta d\varphi = 2\pi\frac{I_0^2\eta}{8\pi^2}\int_{\theta=0}^{\pi}\frac{\left(cos(\pi cos\theta) + 1\right)^2}{sin\theta}d\theta$$
$$= \frac{I_0^2\eta}{4\pi}\int_{\theta=0}^{\pi}\frac{\left(cos(\pi cos\theta) + 1\right)^2}{sin\theta}d\theta$$

The integral is the same as in **Eq. (18.92)** and, from **Table 2**, equals 3.318:

$$P_{rad} = \frac{I_0^2\eta}{4\pi} 3.318 = \frac{1\times 377}{4\pi} 3.318 = 99.54 \qquad [\mathrm{W}]$$

b. From **Eq. (18.36)**:

$$P_{rad} = \frac{I_0^2}{2} R_{rad} \qquad \rightarrow \qquad R_{rad} = \frac{2P_{rad}}{I_0^2} = 2\times 99.54 = 199.08 \qquad [\Omega]$$

c. The definition of directivity is given in **Eq. (18.50)** and that of radiation intensity in **Eq. (18.46)**:

$$D(\theta,\varphi) = \frac{U(\theta,\varphi)}{P_{rad}/4\pi} = \frac{4\pi U(\theta,\varphi)}{P_{rad}}$$

$$U(\theta,\varphi) = P_{av}R^2 = \frac{I_0^2\eta}{8\pi^2}\left(\frac{cos(\pi cos\,\theta) + 1}{sin\,\theta}\right)^2 \qquad \left[\frac{\mathrm{W}}{\mathrm{sr}}\right]$$

Thus:

$$D(\theta,\varphi) = \frac{4\pi U(\theta,\varphi)}{P_{rad}} = \frac{4\pi}{199.08}\frac{I_0^2\eta}{8\pi^2}\left(\frac{cos(\pi cos\theta) + 1}{sin\theta}\right)^2 = 0.60278\left(\frac{cos(\pi cos\theta) + 1}{sin\theta}\right)^2$$

For maximum directivity we note that the maximum value of the expression in brackets occurs at $\theta=\pi/2$ and equals 2. Thus:

$$d = 4\times 0.60278 = 2.412$$

2. A half wavelength antenna is given. The maximum magnetic field intensity of the antenna in the far field is measured and found to be equal to 1 A/m (peak). Assuming free space and no losses, calculate the radiated power of the antenna. Radiation is at a 1m wavelength.

Solution: Given the magnetic field intensity in the far field we can immediately obtain the electric field intensity and therefore the time averaged power density. Integrating over a sphere of radius R gives the radiated power.

Method A:
From **Eq. (18.98):**

$$\mathbf{H} = \hat{\varphi}\frac{jI_0}{2\pi R}e^{-j\beta R}\frac{cos\left(\frac{\pi}{2}cos\theta\right)}{sin\theta} = \hat{\varphi}j\,1e^{-j\beta R}\frac{cos\left(\frac{\pi}{2}cos\theta\right)}{sin\theta} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

The electric field intensity is therefore (**Eq. (18.97)**):

$$\mathbf{E} = \hat{\theta}j\eta e^{-j\beta R}\frac{cos\left(\frac{\pi}{2}cos\theta\right)}{sin\theta} \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

The power density is therefore:

$$\mathbf{P}_{av} = \frac{\mathbf{E}\times\mathbf{H}^*}{2} = \hat{\mathbf{R}}\eta\left[\frac{cos((\pi/2)cos\theta)}{sin\,\theta}\right]^2 \qquad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

The radiated power is:

$$P_{rad} = \frac{\eta}{2}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\frac{cos\left((\pi/2)cos\theta\right)}{sin\,\theta}\right]^2 R^2 sin\,\theta d\theta d\varphi = \pi\eta R^2 \int_{\theta=0}^{\pi}\frac{\left[cos\left((\pi/2)cos\theta\right)\right]^2}{sin\,\theta}d\theta = 1.218\pi\eta R^2 = 1442.57R^2 \quad [\mathrm{W}]$$

Notes:

1. 1.218 is the value of the integral taken from Table 2.
2. The radiated power depends on where the magnetic field intensity was measured.

Method A:

An alternative method is to note from the expression for **H** that

$$\frac{|I_0|}{2\pi R} = 1 \quad \rightarrow \quad |I_0| = 2\pi R$$

Now, since we know that the radiation resistance of a half wavelength dipole is 73.08 W and using Eq. (18.36) we have:

$$P_{rad} = \frac{I_0^2}{2}R_{rad} = \frac{(2\pi R)^2}{2}73.08 = 1442.54R^2 \quad [\mathrm{W}]$$

3. Three half wavelength dipoles are placed parallel to each other and are separated a distance h=1m. The three dipoles are fed with identical currents equal to 2A (peak) at 1.2 GHz. Find:

a. The total radiated power of the array.

b. The directivity of the array.

Solution: Use the expressions for an n element array with n=3.

a. The electric field intensity of a 3 element array of half wavelength dipoles may be written directly from **Eqs. (18.132)** and **(18. 97)**:

$$\mathbf{E} = \hat{\theta}\frac{j\eta I_0}{2\pi R}e^{-j\beta R}\frac{cos\left((\pi/2)cos\theta\right)}{sin\,\theta}e^{j\psi}\frac{sin(3\psi/2)}{sin(\psi/2)}$$

where in this case,

$$\psi = \beta h sin\theta cos\varphi = (2\pi/\lambda)(4\lambda)sin\,\theta cos\varphi = 4\pi sin\theta cos\varphi$$

where the fact that the wavelength equals 0.25m ($\lambda = c/f = 3\times10^8/1.2\times10^9 = 0.25$m) was used to write h=4λ.

Similarly, for the magnetic field intensity we write:

$$\mathbf{H} = \hat{\varphi}\frac{jI_0}{2\pi R}e^{-j\beta R}\frac{cos\left((\pi/2)cos\theta\right)}{sin\,\theta} \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

The time averaged power density is therefore:

$$\mathbf{P}_{av} = \frac{\mathbf{E}\times\mathbf{H}^*}{2} = \hat{\mathbf{R}}\frac{\eta I_0^2}{4\pi^2 R^2}\left[\frac{cos\left((\pi/2)cos\theta\right)}{sin\,\theta}\times\frac{sin(3\psi/2)}{sin(\psi/2)}\right]^2 \quad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

The radiated power is:

$$P_{rad} = \frac{\eta I_0^2}{4\pi^2}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{R^2}\left[\frac{cos((\pi/2)cos\theta)}{sin\,\theta}\times\frac{sin(3\psi/2)}{sin(\psi/2)}\right]^2 R^2 sin\,\theta d\theta d\varphi = \frac{\eta I_0^2}{2\pi}\int_{\theta=0}^{\pi}\frac{\left[cos((\pi/2)cos\theta)sin(3\psi/2)\right]^2}{\left[sin(\psi/2)\right]^2 sin\,\theta}d\theta$$

$$= \frac{\eta I_0^2}{2\pi}\int_{\theta=0}^{\pi}\frac{\left[cos((\pi/2)cos\theta)sin(6\pi sin\theta cos\varphi)\right]^2}{\left[sin(2\pi sin\theta cos\varphi)\right]^2 sin\,\theta}d\theta = 240\int_{\theta=0}^{\pi}\frac{\left[cos((\pi/2)cos\theta)sin(6\pi sin\theta cos\varphi)\right]^2}{\left[sin(2\pi sin\theta cos\varphi)\right]^2 sin\,\theta}d\theta \qquad [\mathrm{W}]$$

b. The definition of directivity is given in **Eq. (18.50)** and that of radiation intensity in **Eq. (18.46)**:

$$D(\theta,\varphi) = \frac{U(\theta,\varphi)}{P_{rad}/4\pi} = \frac{4\pi U(\theta,\varphi)}{P_{rad}}$$

$$U(\theta,\varphi) = P_{av}R^2 \qquad \left[\frac{\mathrm{W}}{\mathrm{sr}}\right]$$

Thus:

$$D(\theta,\varphi) = \frac{4\pi \mathcal{P}_{av}R^2}{P_{rad}} = \frac{4\pi\frac{\eta I_0^2}{4\pi^2}\left[\frac{cos((\pi/2)cos\theta)}{sin\,\theta}\times\frac{sin(6\pi sin\theta cos\varphi)}{sin(2\pi sin\theta cos\varphi)}\right]^2}{240\int_{\theta=0}^{\pi}\frac{\left[cos((\pi/2)cos\theta)sin(6\pi sin\theta cos\varphi)\right]^2}{\left[sin(2\pi sin\theta cos\varphi)\right]^2 sin\,\theta}d\theta} = 2\frac{\left[\frac{cos((\pi/2)cos\theta)}{sin\,\theta}\times\frac{sin(6\pi sin\theta cos\varphi)}{sin(2\pi sin\theta cos\varphi)}\right]^2}{\int_{\theta=0}^{\pi}\frac{\left[cos((\pi/2)cos\theta)sin(6\pi sin\theta cos\varphi)\right]^2}{\left[sin(2\pi sin\theta cos\varphi)\right]^2 sin\,\theta}d\theta}$$

4.

a. Calculate the ratio between the radiation resistance of a *2*λ and a λ*/2* dipole, at any given frequency.

b. Calculate the ratio between the maximum power density of a *2*λ and a λ*/2* dipole.

c. Based on the previous two responses which antenna is “better” overall?. Assume lossless antennas

Solution: Calculate the radiated power for an arbitrary long antenna (**Eq. 18.93**) and evaluate it for a 2λ and a λ/2 antenna. The ratio is then found. In (b), the same relation is used.

a. From **Eq. (18.93)**:

$$R_{rad} = \frac{\eta}{2\pi}\int_{\theta=0}^{\theta=\pi}\frac{\left(cos((L\pi/\lambda)cos\theta) - cos(L\pi/\lambda)\right)^2}{sin\theta}d\theta$$

The integral is given in **Table 2** for L=2λ and for L=λ/2 as 4.327 and 1.218. With these:

$$R_{rad}(\lambda/2) = \frac{1.218\eta}{2\pi}, \qquad R_{rad}(2\lambda) = \frac{4.327\eta}{2\pi}$$

Thus:

$$R_{rad}(\lambda/2) = \frac{1.218\eta}{2\pi}, \qquad \frac{R_{rad}(2\lambda)}{R_{rad}(\lambda/2)} = \left(\frac{4.327\eta}{2\pi}\right)\Big/\left(\frac{1.218\eta}{2\pi}\right) = \frac{4.327}{1.218} = 3.55$$

b. The time averaged power density for an arbitrary long antenna is given in **Eq. (18.89):**

$$|\mathbf{P}_{av}| = \frac{\eta I_0^2}{8\pi^2 R^2}\left(\frac{cos((\beta L/2)cos\theta) - cos(\beta L/2)}{sin\,\theta}\right)^2 \qquad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

For the half wavelength dipole we have:

$$P_{av} = \frac{\eta I_0^2}{8\pi^2 R^2}\left(\frac{\cos((\pi/2)\cos\theta)}{\sin\theta}\right)^2 \qquad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

The maximum value occurs at $\theta=\pi/2$ and equals:

$$P_{av}\big|_{max} = \frac{\eta I_0^2}{8\pi^2 R^2} \qquad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

For the 2λ dipole:

$$\left|\mathbf{P}_{av}\right| = \frac{\eta I_0^2}{8\pi^2 R^2}\left(\frac{\cos(2\pi\cos\theta)-1}{\sin\theta}\right)^2 \qquad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

The maximum value occurs when $\cos(2\pi\cos\theta)=-1$. This occurs at $\theta=60°$. Thus:

$$P_{av}\big|_{max} = \frac{\eta I_0^2}{8\pi^2 R^2}\left(\frac{-1-1}{\sin 60}\right)^2 = \frac{\eta I_0^2}{8\pi^2 R^2}\left(\frac{-2}{\sqrt{3}/2}\right)^2 = 5.333\frac{\eta I_0^2}{8\pi^2 R^2} \qquad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

Thus:

$$\frac{P_{av}(2\lambda)\big|_{max}}{P_{av}(\lambda/2)\big|_{max}} = 5.333$$

c. Based on the information in (a) and (b), the 2λ antenna is much better. However, other aspect come into play including rsadiation resistance, sidelobes and the like.